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Simple Version of the Greenberger–Horne–Zeilinger (GHZ) Argument Against Local Realism

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Here is a simple, clear, useful proof that quantum mechanics contradicts Einstein, Podolsky, and Rosen's local realistic assumptions. It is a variant of the powerful argument first worked out by Daniel Mordechai Greenberger, Michael A. Horne, and Anton Zeilinger. This version uses the eigenstates of two orthogonal spin components for three spin-1/2 particles. No operator or matrix algebra is necessary. A novel discussion of the background and history serves to introduce this proof and to place it in the context of Danny's work.

1. BACKGROUND AND INTRODUCTION

In 1935, Einstein, Podolsky, and Rosen (EPR)[1] introduced local realistic premises into the examination of quantum theory. Their exposition uses four logical assumptions. Two are conditions for an element of reality and for completeness.[2] They stated, "*If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*" And for a theory to be complete, "*every element of the physical reality must have a counterpart in the physical theory*" (original italics).

Two premises supplemented these conditions: (A) that quantum theory implies, for two variables represented by noncommuting operators, there exist values for one variable which entail that the other variable has no definite value (hence is not an element of physical reality); and (B) an

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implicit invocation of locality, to enforce strict noninteraction between two spatially separated systems—this implied that one could measure alternative properties of the first system and "not disturb system in any way"[3] if the two systems were observed at space-like separation.

The actual proof of EPR is, in formal logic terms, a direct proof—not the proof by contradiction announced in their abstract and introduction. EPR wrote down a two-particle state entangled in both position and momentum. The value of each of these variables for the first particle can be determined exactly by local measurement on the second, remote particle. Hence by EPR's condition there exist elements of reality corresponding to both the position and the momentum of each particle. These simultaneous elements of reality correspond to noncommuting operators for which, by premise (A), quantum mechanics cannot give values. Since the definition of completeness of a theory requires it to provide definite values, quantum mechanics cannot be complete. This reasoning never actually uses the assumption of quantum mechanics' completeness to reach its conclusion of incompleteness.[4]

EPR were careful to base their argument on those cases where even the statistical predictions of quantum mechanics produced certainty—probabilities of either 100 or 0%. If quantum theory were correct (which EPR clearly accepted), then these certainties would be reproduced in the complete theory, which they believed was possible. As mentioned in the last paragraph of the EPR paper, they sought such a future theory "to provide complete description of the physical reality."

The seriousness of this quest should not be underestimated. Einstein took nonlocality very seriously and found it a fertile area of research as shown by a publication it the same journal a few months later—the famous paper[5] introducing to general relativity the Einstein–Rosen "bridges," now known as "wormholes." In a remarkable overlooked passage, they propose bridges as essential structures for elementary quantum particles:

> In favor of the [new] theory one can say...that in principle it can claim to be complete (or closed). On the other hand one does not see *a priori* whether the theory contains the quantum phenomena. Nevertheless, one should not exclude *a priori* the possibility that the theory may contain them.[6]

They were serious enough to point out that massive charged particles must contain at least two bridges to exhibit Coulomb *repulsion*, since a single such concentration of space–time curvature could only *attract* a similar (positive energy) structure. It would be dramatic indeed if today's currently important multidimensional strings, 'branes, and M-formalism eventually led to a theory that (in effect) found such essentially quantum phenomona as entanglement emerging from dynamics of a geometrical structure, thus unifying gravity and quantum mechanics—and fulfilling Einstein's 1935 quest for completeness.

Others took the EPR paper seriously, too. Though Bohr's reply[7] never satisfied Einstein, the EPR paper spawned an entire field of physics. Bohm's[8] masterful introduction of the singlet (spin-zero) state of two spin-1/2 particles with discrete variables (in place of EPR's position and momentum) set the stage for Bell and eventually for the surprise of GHZ argument.

John Bell[9] compared EPR realism and locality with the probabilistic quantum predictions of Bohm's state, not just the deterministic ones EPR had explicitly considered. Bell first emphasized the role of spatial separation to ensure EPR noninteraction. Then he proved, surprisingly, that the probabilistic quantum mechanical predictions of Bohm's state are simply unmatchable by any local realistic theory whatsoever. But he needed the extension beyond deterministic cases: Bell's inequality fails to distinguish local realistic theories from quantum mechanics precisely for those cases of perfect predictability that EPR emphasized. Eventually actual experiments[10] were performed, confirming quantum theory not EPR-local realism.

In the late 1980s Dan Greenberger began investigations destined to lead to the GHZ disproof of the existence of elements of reality themselves, using only the cases of deterministic prediction that EPR had emphasized. Instead of the two-particle singlet state favored by Bell, Greenberger looked at the cascade decay of a spin-zero state to two spin-1 particles, each in turn decaying to two spin-1/2 particles.[11]

2. THE GHZ THEOREM—A SIMPLE VERSION

In deriving this result, Greenberger, Horne, and Zeilinger originally considered the perfect correlations among all *four* particles, showing there are *no* possible consistent values for all the EPR elements of reality. However, the paper noted that one particle was a spectator: the proof relied on only three. Following the GHZ publication,[12] Mermin[13] and others[2] simplified the proof.

Here we reconsider the GHZ argument in the case of *three* spin-1/2 particles. The perfect correlations are between outcomes of spin-component measurements along various orthogonal axes: paradigmatic Stern–Gerlach experiments. This proof uses only states, not operators or matrix algebra.

Consider the state

$$|\Psi\rangle = (|\mathrm{UUU}\rangle + |\mathrm{DDD}\rangle)/\sqrt{2} \quad (1)$$

where U and D stand for spin up and down along the z-axis, respectively. The state $|\Psi\rangle$ may be expressed in basis vectors for positive and negative x-spin, $|R\rangle = (|U\rangle + |D\rangle)/\sqrt{2}$ and $|L\rangle = (|U\rangle - |D\rangle)/\sqrt{2}$, where R and L stand for right and left. It can also be expressed in positive and negative eigenstates of y-spin, $|F\rangle = (|U\rangle + i|D\rangle)/\sqrt{2}$ and $|B\rangle = (|U\rangle - i|D\rangle)/\sqrt{2}$, respectively, where F and B stand for forward and backward along the y-axis.

If we elect to use x-spin for the first particle and y-spin for the other two particles, state (1) is

$$|\Psi\rangle = (|RFB\rangle + |RBF\rangle + |LFF\rangle + |LBB\rangle)/2 \tag{2}$$

Note that when the state of the first particle is $|R\rangle$, the other particles always have opposite y-spin, and when it is $|L\rangle$, their y-spins agree. Clearly we can predict the x-spin of a particle without in any way disturbing it, simply by measuring the y-spin of the other two particles. Similarly, the value of y-spin for any of the particles can be determined by measuring the x-spin of another particle *and* the y-spin of the third. Thus the x-spin and y-spin of each particle must both be EPR elements of reality. Moreover, reasoning from their reality, these properties exist even when they are not directly measured.

If we elect to use x-spin for all three particles, then

$$|\Psi\rangle = (|RRR\rangle + |RLL\rangle + |LRL\rangle + |LLR\rangle)/2 \tag{3}$$

Note that each term contains an odd number of positive x-spins. Now consider the three *pairs* of particles (1 and 2, 2 and 3, and 1 and 3) that are present. From Eq. (2) an odd number of these pairs must have opposite y-spins, because positive x-spin implies opposite y-spin values. The $|RRR\rangle$ term implies that all three pairs of particles are opposite in y-spin, and the other three terms in (3) imply that just one pair is. Thus (2) and (3) together imply that there must be an odd number of opposite y-spin pairs. But that is impossible. The three y-spins, as elements of reality, must all agree (as FFF or BBB) or a single value may disagree (all six other cases, e.g., FBB, FBF, etc.). In either case the number of opposite-spin pairs is even!

EPR's reality condition is thus inconsistent with the predictions of quantum mechanics, without any use of statistical predictions. Since quantum mechanics is assumed correct in the EPR paper—and repeatedly confirmed in physics experiments to date—the conjunction of their assumptions regarding reality and locality is inadmissible. Just as EPR, using the correctness of the quantum theory as a foundation, sought to bring physicists to the unwelcome conclusion that the theory's representation of reality is incomplete, Greenberger's GHZ argument uses quantum theory to show that EPR-type local reality cannot be accepted.

What this work shows is that physicists cannot impose their preconceptions on reality or on our theories. We have no “crystal ball” to reveal the surprises physics has in store. No one is better equipped than Danny to study the issues that arise and to enlighten us all with wit and wisdom.

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