matrix with nonnegative integer entries satisfying the row and column sum conditions if and only if

$$r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n.$$

2.7.3. *Ryser's proof.*[54]

Let m and n be given positive integers, $(r_1, r_2, \ldots, r_m)$ be a given row sum vector, $\underline{r}^*$ be a length-n conjugate of $\underline{r}$, and (a_{ij}) be the $m \times n$ Ferrers matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } j \le r_i, \\ 0 & \text{otherwise.} \end{cases}$$

Show that if $(s_1, s_2, \ldots, s_m) \preceq \underline{r}^*$, then one can rearrange the 1s in (a_{ij}) into a matrix with row sums $\underline{r}$ and column sums $\underline{s}$ by *interchanges* of the form

$$\begin{matrix} 1 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 1 \end{matrix} \quad \leftrightarrow \quad \begin{matrix} 0 & \ldots & 1 \\ \vdots & & \vdots \\ 1 & \ldots & 0 \end{matrix}$$

Show the more general result: let A and B be two (0-1)-matrices with row sums $\underline{r}$ and column sums $\underline{s}$. Then A can be transformed into B by interchanges.

2.7.4. *A submodular function proof.*

Define the *excess function* $\eta(A)$ by the formula

$$\eta(A) = |R(A)| - \left[\sum_{i=1}^{k} (|A| - d_i)^+ \right].$$

(a) Show that η is submodular.

(b) Prove Higgins' theorem by induction, using the proof of the matroid marriage theorem as a guide.

2.7.5. (Research problem) Find an analog of the Gale–Ryser theorem for symmetric (0-1)-matrices.

2.7.6. (Research problem) *An extension of Birkhoff's theorem.*

Let $(r_1, r_2, \ldots, r_m)$ and $(s_1, s_2, \ldots, s_n)$ be vectors with nonnegative integer entries. The set Matrix$(\underline{r}, \underline{s})$ of all $m \times n$ matrices with real nonnegative entries with row sums $\underline{r}$ and column sums $\underline{s}$ is convex. Determine the extreme points of Matrix$(\underline{r}, \underline{s})$.

[54] Ryser (1957).

2.8 Matching Theory in Higher Dimensions

"The possibility of extending the marriage theorem to several dimensions does not seem to have been explored. Thinking rather crudely, one might replace a matrix by a tensor." In this section, we explain this remark, made by Harper and Rota in *Matching theory* (p. 211).

We begin with some *informal* tensor algebra. Let $V_1, V_2, \ldots, V_k$ be vector spaces of dimension $d_1, d_2, \ldots, d_k$ over a field $\mathbb{F}$. A *decomposable k-tensor* in the *tensor product* $V_1 \otimes V_2 \otimes \cdots \otimes V_k$ is a formal product $v_1 \otimes v_2 \otimes \cdots \otimes v_k$, where the ith vector v_i is in V_i. A *k-tensor* is a linear combination of decomposable k-tensors. Tensors are *multilinear*; that is, they satisfy the property

$$\begin{aligned} v_1 &\otimes \cdots \otimes (\lambda v_i + \mu u_i) \otimes \cdots \otimes v_m \\ &= \lambda(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_m) \\ &\quad + \mu(v_1 \otimes \cdots \otimes u_i \otimes \cdots \otimes v_m) \end{aligned}$$

and all relations implied by this property.

A *covector* or *dual vector* of V is a linear functional or a linear transformation $V \to \mathbb{F}$. The covectors form the vector space V^* dual to V. Let $e_1, e_2, \ldots, e_d$ be a chosen basis for V. Then $e_1^*, e_2^*, \ldots, e_d^*$ is the basis for the dual V_i^* defined by $e_i^*(e_j) = 0$ if $i \neq j$ and 1 if $i = j$; in other words, $e_i^*(e_j) = \delta_{ij}$.

Consider the tensor product $\mathbb{F}^m \otimes (\mathbb{F}^n)^*$. The tensor $e_i \otimes e_j^*$ defines a linear transformation $\mathbb{F}^n \to \mathbb{F}^m$ by $u \mapsto e_j^*(u)e_i$. The matrix of this linear transformation has ij-entry equal to 1 and all other entries equal to 0. Hence, the matrix (a_{ij}) can be regarded as the tensor

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} e_i \otimes e_j^*.$$

Generalizing this, and choosing bases for V_i, we can think of a k-tensor as a k-dimensional array of numbers from $\mathbb{F}$.

2.8.1. Lemma. The rank of a matrix A equals

$$\min \left\{ s\colon A = \sum_{i=1}^{s} v_i \otimes u_i^* \right\},$$

the minimum number of decomposable 2-tensors in an expression of A as a sum of decomposable 2-tensors.

Proof. Suppose $\sum_{i=1}^{s} v_i \otimes u_i^*$. Then the image of A is spanned by the vectors $v_1, v_2, \ldots, v_s$. Hence, the rank of A, which equals the dimension of the image of A, is at most s.

Let A be an $m \times n$ matrix of rank ρ. Let $r_1, r_2, \ldots, r_\rho$ be ρ linearly independent rows in A, and

$$r_i = \sum_{j=1}^{\rho} b_{ij} r_j, \text{ for } \rho + 1 \leq i \leq m.$$

Then A can be written as a sum

$$\left(e_1 + \sum_{h=\rho+1}^{m} b_{h1} e_h\right) \otimes r_1^* + \left(e_2 + \sum_{h=\rho+1}^{m} b_{h2} e_h\right) \otimes r_2^* + \cdots + \left(e_\rho + \sum_{h=\rho+1}^{m} b_{h1} e_h\right) \otimes r_\rho^*$$

of s decomposable tensors. Hence, $\rho \geq s$. □

An example might make the second part of the proof clearer. Consider the matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 6 & 7 & 1 & 9 \end{pmatrix}.$$

Then

$$r_1^* : (x_1, x_2, x_3, x_4) \mapsto x_2 + 2x_3 + 3x_4 \quad \text{and}$$
$$r_2^* : (x_1, x_2, x_3, x_4) \mapsto 2x_1 + 3x_2 + x_3 + 5x_4,$$

and the matrix can be written as the sum of two decomposable tensors by the following computation:

$$e_1 \otimes r_1^* + e_2 \otimes r_2^* + e_3 \otimes (-2r_1^* + 3r_2^*) = (e_1 - 2e_3) \otimes r_1^* + (e_2 + 3e_3) \otimes r_2^*.$$

Motivated by Lemma 2.8.1, we define the *rank* of a tensor as the minimum number of decomposable tensors in an expression of A as a sum of decomposable tensors.[55] In the case of matrices, the rank can be calculated (efficiently) by triangulation or Gaussian elimination. There is no known analogous algorithm for tensors. For matrices, there is also the determinant, which determines whether a square matrix has full rank. There is no invariant for tensors which is as explicit or useful as the determinant for matrices. To oversimplify, one

[55] This is the commonly accepted definition of the rank of a tensor for algebraists.

of primary aims of "classical invariant theory" is to find concepts analogous to triangulations and determinants for general tensors.[56] The research problem here is to develop an elegant (and hence useful) theory. Whatever this theory is, it should contain a higher-dimensional matching theory as a special case.

Let $A \subseteq \{1, 2, \ldots, d_1\} \times \{1, 2, \ldots, d_2\} \times \cdots \times \{1, 2, \ldots, d_k\}$. The *free tensor* supported by A is the k-tensor

$$\sum_{\{i_1, i_2, \ldots, i_k\} \subseteq A} x_{i_1, i_2, \ldots, i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$$

in $\mathbb{F}^{d_1} \otimes \mathbb{F}^{d_2} \otimes \cdots \otimes \mathbb{F}^{d_k}$, where $\mathbb{F}$ is a sufficiently large field so that the nonzero entries $x_{i_1, i_2, \ldots, i_k}$ are algebraically independent over some given subfield of $\mathbb{F}$. The problem of higher-dimensional matching theory is to define, for a given free tensor, a combinatorial object whose "size" equals its rank. Such a definition should indicate the right explicit definition for the determinant of a tensor. Optimistically, one would also hope for an analog of the König–Egerváry theorem, where the rank equals the minimum over some "cover" of A.

Related to this problem is the question of higher-dimensional submodular inequalities (see Section 2.4). We will need to assume knowledge of matroid theory. A main axiom of matroid theory is the submodular inequality for the rank function rk:

$$\text{rk}(A) + \text{rk}(B) \geq \text{rk}(A \cup B) + \text{rk}(A \cap B).$$

This inequality is satisfied by the rank function of vectors, that is, 1-tensors. With an easy argument (see Exercise 2.8.1), this inequality can be adapted to matrices or 2-tensors. If $M[T|S]$ is a matrix, let $M[B|A]$ be the submatrix obtained by restricting M to the rows B and columns A and $\text{rank}(B, A)$ be the rank of the submatrix $M[B|A]$. Then

$$\text{rank}(B, A) + \text{rank}(D, C) \geq \text{rank}(B \cup D, A \cap C) + \text{rank}(B \cap D, A \cup C).$$

This inequality is the *bimatroid submodular inequality*.[57] It compares ranks of *rectangular* sets of the form $B \times A$. There is no known analog of the submodular inequality for higher-dimensional tensors. Indeed, no "new" rank inequality for higher tensors is known. This would be a step toward a higher-dimensional matroid theory.

[56] See Grosshans (2003) and Rota (2002); for a mainstream account, see Gelfand et al. (1994).
[57] Kung (1978) and Schrijver (1979).

Exercises

2.8.1. (a) Prove the bimatroid submodular inequality.

(b) (Research problem) The bimatroid inequality involves "rectangular" sets of the form $B \times A$. Are there rank inequalities involving arbitrary subsets of $T \times S$ for free matrices?

2.8.2. *Rank inequalities for vectors.*[58]

(a) Prove that the rank function rk of vectors satisfies *Ingleton's inequality*: for four subsets X_1, X_2, X_3, and X_4 of vectors,

$$\begin{aligned} &\mathrm{rk}(X_1) + \mathrm{rk}(X_2) + \mathrm{rk}(X_1 \cup X_2 \cup X_3) + \mathrm{rk}(X_1 \cup X_2 \cup X_4) + \mathrm{rk}(X_3 \cup X_4) \\ &\quad \le \mathrm{rk}(X_1 \cup X_2) + \mathrm{rk}(X_1 \cup X_3) + \mathrm{rk}(X_1 \cup X_4) + \mathrm{rk}(X_2 \cup X_3) \\ &\qquad + \mathrm{rk}(X_2 \cup X_4). \end{aligned}$$

Show that this inequality does not follow from the submodular inequality.

(b) (Research problem posed by A. W. Ingleton) Find other inequalities for rank functions of vectors.

(c) (Research problem) "Describe" all the inequalities satisfied by rank functions of vectors. (It is known that the set of forbidden minors for representability over the real or complex numbers is infinite. However, this does not preclude a reasonable description of all rank inequalities. For example, are all rank equalities for vectors consequences of Grassmann's equality?)

2.8.3. *Rank inequality for matrices.*

There are many matrix rank inequalities involving products of matrices. The simplest is

$$\mathrm{rank}(AB) \le \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}.$$

Another is the Frobenius rank inequality

$$\mathrm{rank}(AB) + \mathrm{rank}(BC) \le \mathrm{rank}(B) + \mathrm{rank}(ABC).$$

Develop a theory of matrix rank inequalities. For example, are they all consequences of a finite set of inequalities?

2.8.4. *Higher-dimensional permanents.*[59]

Let A be an $n_1 \times n_2 \times \cdots \times n_d$ array of numbers with entries $a_{i_1, i_2, \ldots, i_d}$. Then a reasonable definition of the *permanent* per A is

$$\mathrm{per}\, A = \sum \prod_{i=1}^{n_1} a_{i, \sigma_1(i), \sigma_2(i), \ldots, \sigma_d(i)},$$

[58] Ingleton (1971). [59] Dow and Gibson (1987) and Muir and Metzler (1933, chapter 1).

where the sum ranges over all d-tuples $(\sigma_1, \sigma_2, \ldots, \sigma_d)$ of one-to-one functions $\sigma_i : \{1, 2, \ldots, n_1\} \to \{1, 2, \ldots, n_i\}$. Extend as many of the properties of two-dimensional permanents as possible. In particular, prove an analog of Exercise 2.6.16(a).

2.9 Further Reading

There are probably as many approaches to matching theory as there are areas of mathematics. The survey paper *Matching theory* shows some of these connections. Approaches we have completely ignored are those of graph theory, combinatorial optimization, polyhedral combinatorics, probabilistic and asymptotic combinatorics, and analysis of algorithms. For further reading, we recommend the following books or survey papers:

R. Brualdi and H.J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, Cambridge, 1991.

R. Brualdi and B.L. Shader, *Matrices of Sign-Solvable Linear Systems*, Cambridge University Press, Cambridge, 1995.

G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, 2nd edition, Cambridge University Press, Cambridge, 1952.

L. Lovász and M.D. Plummer, *Matching Theory*, North-Holland, Amsterdam and New York, 1986.

A.W. Marshall and I. Olkins, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York and London, 1979.

H. Minc, *Permanents*, Addison-Wesley, Reading, MA, 1978.

H. Minc, *Nonnegative Matrices*, Wiley, New York, 1988.

L. Mirsky, Results and problems in the theory of doubly-stochastic matrices, *Z. Wahrscheinlichkeittheor. Verwandte Geb.* 1 (1962–1963) 319–334.

L. Mirsky, *Transversal Theory*, Academic Press, New York, 1971.