TRACE IDENTITIES FROM IDENTITIES FOR DETERMINANTS

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Abstract. We present new identities for determinants of matrices $(A_{i,j})$ with entries $A_{i,j}$ equal to $a_{i,j}$ or $a_{i,0}a_{0,j} - a_{i,j}$, where the $a_{i,j}$'s are indeterminates. We show that these identities are behind trace identities for $SL(2,\mathbb{C})$ matrices found earlier by Magnus in his study of trace algebras.

1. Introduction. In this paper we derive an infinite family of new trace identities for 2×2 matrices by using an infinite family of new determinantal identities. These trace identities generalise a certain trace identity due to Magnus.

Trace identities for 2×2 matrices have been studied for over 100 years, one of the original motivations being the investigation of Teichmüller space via representations of surface groups as (certain equivalence classes of) subgroups of $SL(2,\mathbb{C})$. This approach originated with Fricke and Klein [1] and there have been many subsequent attempts at ways of giving real analytic trace coordinates for Teichmüller space (see for example [2, 3, 5, 7] and references therein).

Actions of groups on trace algebras have been investigated by Vogt [10] and more recently by Magnus [4]. Vogt was interested in studying invariants of differential equations, while Magnus was concerned with automorphisms and outer automorphisms of free groups. Physicists have also taken an interest in trace relations [6]. One thus sees the variety of applications that these ideas have.

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In [4], Magnus's penultimate paper, he investigated the action of the automorphism group $Aut(F_n)$ of a free group F_n of rank n on the traces of generic 2×2 matrices. These generic traces generate an algebra Q_n as follows: let $m_1,\dots,m_n$ be 'generic' matrices in $SL(2,\mathbb{Z})$ and for any sequence of distinct elements $i_1<i_2<\dots<i_k$, $k\le n$, of $\{1,2,\dots,n\}$ we let

$$\tau_{i_1 i_2\dots i_k}(m_1,m_2,\dots,m_n)=\operatorname{tr}(m_{i_1}m_{i_2}\cdots m_{i_k}),$$

where tr denotes the trace function. Then there are certain relations among traces of 2×2 matrices that show that the $\tau_i,\tau_{jk},\dots$ generate the algebra of all the traces of elements of the group $\langle m_1,\dots,m_n\rangle$: Take a polynomial algebra Q_n' generated by independent indeterminates $\tau_i',\tau_{jk}',\dots$; let $I\subset Q_n'$ denote the ideal of all elements $r(\tau_i',\tau_{jk}',\dots)\in Q_n'$ such that

$$r(\tau_i(m_1,\dots,m_n),\tau_{jk}(m_1,\dots,m_n),\dots)=0$$

for all choices of m_i. Then $Q_n=Q_n'/I$.

In [4] Magnus was concerned with obtaining representations of $Out(F_n)=Aut(F_n)/Inn(F_n)$ by looking at an induced action of $Out(F_n)$ on the trace algebra Q_n. In order to do this he first investigates Q_n. In doing this Magnus constructs various remarkable formulae satisfied by the generators of the trace algebra. These are expressed as equations in the determinants of certain matrices whose entries are traces of elements of $SL(2,\mathbb{C})$. He calls these the *general identities* [4; p. 94] and uses them to derive relations in the trace algebra that are needed for the proof that Q_n is finitely generated and other relevant properties.

These general identities are described as follows:

Magnus's Main Lemma. *For $m_i,M_j\in SL(2,\mathbb{C})$, $1\le i,j\le 4$, we have the following trace relations:*

$$\begin{aligned}&\det(\operatorname{tr} m_iM_j)+\det(\operatorname{tr} m_iM_j^{-1})=0;\\&\det(\operatorname{tr} m_im_j)\det(\operatorname{tr} M_iM_j)=[\det(\operatorname{tr} m_iM_j)]^2.\end{aligned}$$

The goal of this paper is to search for the intrinsic background of trace identities of the above kind. As a first observation, if $m_i,M_j\in SL(2,\mathbb{C})$, $0\le i,j$, with $m_0=M_0=I_2$, the 2×2 identity matrix, and if we write $a_{i,j}=\operatorname{tr} m_iM_j$, so that, in particular, $a_{i,0}=\operatorname{tr} m_i$ and $a_{0,j}=\operatorname{tr} M_j$ for $i,j\ge 1$, then a simple calculation shows that

$$\operatorname{tr} m_iM_j^{-1}=a_{i,0}a_{0,j}-a_{i,j}.$$

Thus, we are led to search for identities involving determinants of matrices $(A_{i,j})$, in which the entries $A_{i,j}$ may be of the form $a_{i,j}$ or $a_{i,0}a_{0,j}-a_{i,j}$. We discovered that on this abstract level there are in fact, somewhat surprisingly, several of these. We summarise our findings in Theorems 1, 3 and 6 in the subsequent sections.

As a corollary to Theorem 1 and to the trace theorem Theorem 2, we obtain the following generalisation of the first of Magnus's formulae.

Magnus's Main Lemma — Generalised. *For* $n \ge 1$ *let*

$$m_1, \dots, m_n, M_1, \dots, M_n \in SL(2, \mathbb{C}),$$

and put $m_0 = M_0 = I_2$. *Define the* $(n+1) \times (n+1)$ *matrix* $A = (A_{i,j})_{0 \le i,j \le n}$ *by*

$$A_{i,j} = \begin{cases} \operatorname{tr}(m_i M_j^{-1}) & \text{if } i+j \text{ is even;} \\ \operatorname{tr}(m_i M_j) & \text{otherwise,} \end{cases}$$

and define $n \times n$ *matrices* $B = (B_{i,j})_{1 \le i,j \le n}$, $C = (C_{i,j})_{1 \le i,j \le n}$ *by* $B_{i,j} = -\operatorname{tr} m_i M_j$, *and* $C_{i,j} = \operatorname{tr} m_i M_j^{-1}$. *Then*

$$\det A = (-1)^n \det B + \det C. \tag{1}$$

Further, if $n \ge 4$, *then* $\det A = 0$, *while if* $n > 4$, *then* $\det B = \det C = 0$.

Theorem 1 implies (1), while the assertion in the last line follows from Theorem 2.

Magnus's first identity is the above result in the situation $n = 4$, the first case where $\det A = 0$, $\det B \ne 0$ and $\det C \ne 0$..

In the next section, we state and prove Theorems 1 and 2. In Section 3, we state and prove a general determinant formula (see Theorem 3), which implies new trace identities for traces of the form $\operatorname{tr} m_i m_j$ and $\operatorname{tr} m_i m_j^{-1}$, where $m_1, m_2, \dots, m_n$ are given matrices in $SL(2, \mathbb{C})$. (That is, they address the case where the second matrix family $M_1, M_2, \dots, M_n$ in our generalisation of Magnus's Main Lemma is identical with the first one.) We close this section with a curious result (see Theorem 6) stating that, in the "skew" case, the determinant of one of the matrices involved in Theorem 3 factors into the product of the "even" part of a Pfaffian and the "odd" part of a Pfaffian, up to a multiplicative constant. (See the paragraph before Theorem 6 for detailed explanations.)

Except for Theorem 2, which is not an abstract determinant identity but a determinant identity specific for traces, we prove our determinant identities by a combinatorial approach, as proposed in [11] (see also [8, Ch. 4]), that is, we combinatorially expand both sides of our identities, and then we bijectively identify the terms on the two sides, possibly helped by an involution which cancels several terms on one side.

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2. The determinant identities which imply the generalisation of Magnus's formula. In this section, we prove a general determinant identity which implies our generalisation (1) of Magnus's formula (see Theorem 1 below), and a general assertion about the vanishing of determinants formed out of traces (see Theorem 2 below) that implies the last assertion in our generalisation of Magnus's Main Lemma, but, in addition, produces many more trace identities.

Theorem 1. *Let $(a_{i,j})_{0\le i,j\le n}$ be a doubly indexed sequence with the property that $a_{0,0}=2$. We let A be the $(n+1)\times(n+1)$ matrix $A=(A_{i,j})_{1\le i,j\le n}$, where*

$$A_{i,j}=\begin{cases}\lambda a_{0,j} & \text{if } 0=i\ne j,\\ a_{i,j} & \text{if } j=0 \text{ or if } i+j \text{ is even},\\ \lambda a_{i,0}a_{0,j}-a_{i,j} & \text{otherwise.}\end{cases}$$

Furthermore, we define two $n\times n$ matrices $B=(B_{i,j})_{1\le i,j\le n}$ and $C=(C_{i,j})_{1\le i,j\le n}$ by

$$B_{i,j}=\lambda a_{i,0}a_{0,j}-a_{i,j},$$

and

$$C_{i,j}=a_{i,j}.$$

Then

$$\det A-(-1)^n\det B-\det C=0. \tag{2}$$

Remark. For better clarity, we remark that, by our definitions, the first row of A is

$$(2,\lambda a_{0,1},\lambda a_{0,2},\dots,\lambda a_{0,n}),$$

while the first column is

$$(2,a_{1,0},a_{2,0},\dots,a_{n,0})^t.$$

For example, for $n=4$, the matrix A is equal to

$$\begin{pmatrix} 2 & \lambda a_{0,1} & \lambda a_{0,2} & \lambda a_{0,3} & \lambda a_{0,4}\\ a_{1,0} & a_{1,1} & \lambda a_{1,0}a_{0,2}-a_{1,2} & a_{1,3} & \lambda a_{1,0}a_{0,4}-a_{1,4}\\ a_{2,0} & \lambda a_{2,0}a_{0,1}-a_{2,1} & a_{2,2} & \lambda a_{2,0}a_{0,3}-a_{2,3} & a_{2,4}\\ a_{3,0} & a_{3,1} & \lambda a_{3,0}a_{0,2}-a_{3,2} & a_{3,3} & \lambda a_{3,0}a_{0,4}-a_{3,4}\\ a_{4,0} & \lambda a_{4,0}a_{0,1}-a_{4,1} & a_{4,2} & \lambda a_{4,0}a_{0,3}-a_{4,3} & a_{4,4}\end{pmatrix}.$$

In view of this remark, it should be clear that the case $\lambda=1$ of this theorem implies (1).

Proof. The first observation is that, in $\det A$, the coefficient of λ^m is zero for $m\ge 2$, and the same is true for $\det B$ and $\det C$. Clearly, this is trivial for $\det C$, which contains no λ at all. To see the claim for $\det B$, we let B' be the matrix $(\lambda a_{i,0}a_{0,j})_{1\le i,j\le n}$ and B'' the matrix $(-a_{i,j})_{1\le i,j\le n}$. Then we expand $\det B=\det(B'+B'')$ in the following form,

$$\det B=\sum_{J\subseteq\{1,2,\dots,n\}}\det B^J,$$

where B^J denotes the matrix where the columns indexed by elements from J are those from B', while the remaining columns are those from B''. Since any two columns from B' are dependent, we have $\det B^J=0$ whenever $|J|\ge 2$. Thus the largest exponent of λ in the expansion of $\det B$ is 1.

For $\det A$ we proceed in the same way. We let A' be the matrix which contains the "λ-terms" from A, and we let A'' be the "rest". To be precise,

$$A'_{i,j} = \begin{cases} \lambda a_{0,j} & \text{if } 0 = i \neq j, \\ 0 & \text{if } j = 0 \text{ or if } i + j \text{ is even, ,} \\ \lambda a_{i,0} a_{0,j} & \text{otherwise,} \end{cases}$$

while

$$A''_{i,j} = \begin{cases} 0 & \text{if } 0 = i \neq j, \\ a_{i,j} & \text{if } j = 0 \text{ or if } i + j \text{ is even,} \\ -a_{i,j} & \text{otherwise.} \end{cases}$$

Again, we have chosen A' and A'' so that $A = A' + A''$. Then we do the same expansion as before,

$$\det A = \sum_{J \subseteq \{1,2,\ldots,n\}} \det A^J,$$

with the analogous meaning of A^J. Again, we have $\det A^J = 0$ if $|J| \geq 2$, this time because the 0-th column of A' is 0, and because any two columns of A' indexed by $j_1, j_2 \geq 1$ which have the same parity are dependent, while if j_1 and j_2 have different parity, $1/\lambda a_{0,j_1}$ times the j_1-st column plus $1/\lambda a_{0,j_2}$ times the j_2-nd column gives the 0-th column of A''.

It remains to verify that the coefficients of λ^0 and of λ^1 in (2) vanish.

Let us begin with the coefficient of λ^0. If we set $\lambda = 0$ in (2), then $\det A$ can be reduced to

$$2 \det((-1)^{i+j} a_{i,j})_{1 \leq i,j \leq n} = 2 \det(a_{i,j})_{1 \leq i,j \leq n} = 2 \det C,$$

while

$$\det B = \det(-a_{i,j})_{1 \leq i,j \leq n} = (-1)^n \det C.$$

Thus, the coefficient of λ^0 in (2) is indeed zero.

For the coefficient of λ^1 we only have to look at $\det A$ and $\det B$. We shall derive combinatorial expressions for these two determinants. In order to do so, let us write $\mathfrak{S}_n$ for the symmetric group on $\{1, 2, \ldots, n\}$. From the definition of the determinant, we have

$$\det B = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn} \sigma \prod_{i=1}^{n} B_{i,\sigma(i)}.$$

Extracting the coefficient of λ^1, we see that the coefficient of λ^1 in $\det B$ is

$$\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}\sigma\sum_{k=1}^{n}a_{k,0}a_{0,\sigma(k)}\prod_{\substack{i=1\\i\neq k}}^{n}(-a_{i,\sigma(i)})$$

$$=\sum_{k=1}^{n}\sum_{l=1}^{n}(-1)^{k+l}a_{k,0}a_{0,l}\sum_{\sigma\in\mathfrak{S}_n^{(k,l)}}\operatorname{sgn}\sigma\prod_{\substack{i=1\\i\neq k}}^{n}(-a_{i,\sigma(i)})$$

$$=(-1)^{n-1}\sum_{k=1}^{n}\sum_{l=1}^{n}(-1)^{k+l}a_{k,0}a_{0,l}\sum_{\sigma\in\mathfrak{S}_n^{(k,l)}}\operatorname{sgn}\sigma\prod_{\substack{i=1\\i\neq k}}^{n}a_{i,\sigma(i)},$$

where $\mathfrak{S}_n^{(k,l)}$ is the set of bijections from $\{1,2,\dots,n\}\backslash\{k\}$ to $\{1,2,\dots,n\}\backslash\{l\}$, where $\operatorname{sgn}\sigma$ has the obvious meaning when identifying $\mathfrak{S}_n^{(k,l)}$ with $\mathfrak{S}_{n-1}$.

Aplying a similar procedure to $\det A$, we obtain that the coefficient of λ^1 in $\det A$ is

$$2\sum_{\sigma\in\mathfrak{S}_n}\sum_{\substack{k=1\\k+\sigma(k)\text{ odd}}}^{n}(\operatorname{sgn}\sigma)\cdot a_{k,0}a_{0,\sigma(k)}\prod_{\substack{i=1\\i\neq k}}^{n}(-1)^{i+\sigma(i)}a_{i,\sigma(i)}$$

$$+\sum_{k=1}^{n}\sum_{l=1}^{n}(-1)^{k+l-1}a_{k,0}a_{0,l}\sum_{\sigma\in\mathfrak{S}_n^{(k,l)}}\operatorname{sgn}\sigma\prod_{\substack{i=1\\i\neq k}}^{n}(-1)^{i+\sigma(i)}a_{i,\sigma(i)}$$

$$=2\sum_{\substack{k,l=1\\k+l\text{ odd}}}^{n}a_{k,0}a_{0,l}\sum_{\sigma\in\mathfrak{S}_n^{(k,l)}}\operatorname{sgn}\sigma\prod_{\substack{i=1\\i\neq k}}^{n}a_{i,\sigma(i)}-\sum_{k=1}^{n}\sum_{l=1}^{n}a_{k,0}a_{0,l}\sum_{\sigma\in\mathfrak{S}_n^{(k,l)}}\operatorname{sgn}\sigma\prod_{\substack{i=1\\i\neq k}}^{n}a_{i,\sigma(i)}$$

$$=\sum_{k=1}^{n}\sum_{l=1}^{n}(-1)^{k+l-1}a_{k,0}a_{0,l}\sum_{\sigma\in\mathfrak{S}_n^{(k,l)}}\operatorname{sgn}\sigma\prod_{\substack{i=1\\i\neq k}}^{n}a_{i,\sigma(i)}.$$

Thus, indeed, the coefficient of λ^1 in (2) is zero.

This completes the proof of the theorem. $\square$

The last assertion of our generalisation of Magnus's Main Lemma follows from the following more general result which gives more trace relations.

Theorem 2. *Let $m_1,m_2,\dots,m_n,M_1,M_2,\dots,M_n\in SL(2,\mathbb{C})$ and let $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n\in\{\pm1\}$. Define the $n\times n$ matrix $D=(D_{i,j})_{1\le i,j\le n}$ by $D_{i,j}=\operatorname{tr}(m_iM_j^{\varepsilon_i})$. If $n\ge 5$, then* $\det D=0$.

Proof. The proof will follow by exhibiting a 0-eigenvector for D.

Fix $j\le n$ and let $M_j=\begin{pmatrix}M_{j1}&M_{j2}\\M_{j3}&M_{j4}\end{pmatrix}$, so that $M_j^{-1}=\begin{pmatrix}M_{j4}&-M_{j2}\\-M_{j3}&M_{j1}\end{pmatrix}$. We also let $m_i=\begin{pmatrix}m_{i1}&m_{i2}\\m_{i3}&m_{i4}\end{pmatrix}$ for all $i\le n$. We will find non-trivial functions $v_1,\dots,v_n$

of the variables $m_{i,j}$ such that $v = (v_1, \dots, v_n)$ is a (left) 0-eigenvector for D, so that $vD = 0$. This will be the case if for all $j \le n$ we have

$$\sum_{i=1}^{n} v_i \operatorname{tr}(m_i M_j^{\varepsilon_i}) = 0.$$

But each of the above equations is linear in the variables $M_{j1}, M_{j2}, M_{j3}, M_{j4}$ and the equations that we so obtain are independent of the column index j. We thus have $n > 4$ linear equations in 4 unknowns. There is thus, generically, a non-trivial solution. Thus, $\det D = 0$ except on a set of measure zero; but since det is continuous it follows that we always have $\det D = 0$. □

3. More determinant identities. Here, we present an identity for determinants of matrices, in which the entries are $a_{i,j}$ or $a_i a_j - a_{i,j}$. Interpreting a_i as the trace of a matrix $m_i \in SL(2, \mathbb{C})$ and $a_{i,j}$ as $\operatorname{tr} m_i m_j$, these identities produce therefore trace identities for the traces $\operatorname{tr} m_i m_j$ and $\operatorname{tr} m_i m_j^{-1}$. In fact, in the theorem below, we allow two additional parameters, λ and β. By specialising them in different ways, we obtain various new determinant identities. The specialisation in Corollary 4 contains the identity which is relevant to the trace case, whereas Corollary 5 contains a "skew" variation. (The "skew" refers to the fact that the matrix A there is skew-symmetric.) As an aside, we prove in Theorem 6 the curious fact, that, in the "skew" case, the determinant of the matrix C factors into two big factors, one of which collects the "even terms" of the Pfaffian of A, the other collecting its odd terms.

We alert the reader that, when compared to Theorem 1, the two theorems below follow a different index convention in that the entries of the matrix A are indexed by i and j from $\{1, 2, \dots, n\}$ (rather than $\{0, 1, \dots, n\}$), and, similarly, the entries of the matrices B and C are indexed by i and j from $\{2, 3, \dots, n\}$ (rather than $\{1, 2, \dots, n\}$). This convention has advantages over the other in the formulation of Theorem 6. A further change of convention is that the "trace-like" entries are $a_{i,j} - \lambda a_{1,i} a_{1,j}$ (rather than $\lambda a_{1,i} a_{1,j} - a_{i,j}$). This allows a more elegant formulation of the following theorem, but, clearly, by multiplying every other row and column of B and C by -1, we could pass to the convention for the "trace-like" entries which is followed in Theorem 1.

Theorem 3. *Let $(a_{i,j})_{1 \le i,j \le n}$ be a doubly indexed sequence with the property that $a_{i,1} = \beta a_{1,i}$ for $i > 1$. We let A be the $n \times n$ matrix $A = (A_{i,j})_{1 \le i,j \le n}$, where*

$$A_{i,j} = \begin{cases} \lambda a_{1,j} & \text{if } 1 = i \ne j, \\ a_{i,j} & \text{otherwise.} \end{cases}$$

Furthermore, we define two $(n-1) \times (n-1)$ matrices $B = (B_{i,j})_{2 \le i,j \le n}$ and $C = (C_{i,j})_{2 \le i,j \le n}$ by

$$B_{i,j} = \begin{cases} a_{i,j} - \lambda a_{1,i} a_{1,j} & \text{if } i + j \text{ is even,} \\ a_{i,j} & \text{if } i + j \text{ is odd,} \end{cases}$$

and

$$C_{i,j} = \begin{cases} a_{i,j} & \text{if } i + j \text{ is even,} \\ a_{i,j} - \lambda a_{1,i} a_{1,j} & \text{if } i + j \text{ is odd.} \end{cases}$$

Then

$$\det A - \beta(\det B + \det C) = (a_{1,1} - 2\beta)\det(a_{i,j})_{2\le i,j\le n}. \tag{3}$$

Remark. For the benefit of the reader, we display the matrices A, B, and C for $n = 5$:

$$A = \begin{pmatrix} a_{1,1} & \lambda a_{1,2} & \lambda a_{1,3} & \lambda a_{1,4} & \lambda a_{1,5} \\ \beta a_{1,2} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ \beta a_{1,3} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ \beta a_{1,4} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \\ \beta a_{1,5} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} \end{pmatrix},$$

$$B = \begin{pmatrix} a_{2,2} - \lambda a_{1,2}^2 & a_{2,3} & a_{2,4} - \lambda a_{1,2}a_{1,4} & a_{2,5} \\ a_{3,2} & a_{3,3} - \lambda a_{1,3}^2 & a_{3,4} & a_{3,5} - \lambda a_{1,3}a_{1,5} \\ a_{4,2} - \lambda a_{1,2}a_{1,4} & a_{4,3} & a_{4,4} - \lambda a_{1,4}^2 & a_{4,5} \\ a_{5,2} & a_{5,3} - \lambda a_{1,3}a_{1,5} & a_{5,4} & a_{5,5} - \lambda a_{1,5}^2 \end{pmatrix},$$

$$C = \begin{pmatrix} a_{2,2} & a_{2,3} - \lambda a_{1,2}a_{1,3} & a_{2,4} & a_{2,5} - \lambda a_{1,2}a_{1,5} \\ a_{3,2} - \lambda a_{1,2}a_{1,3} & a_{3,3} & a_{3,4} - \lambda a_{1,3}a_{1,4} & a_{3,5} \\ a_{4,2} & a_{4,3} - \lambda a_{1,3}a_{1,4} & a_{4,4} & a_{4,5} - \lambda a_{1,4}a_{1,5} \\ a_{5,2} - \lambda a_{1,2}a_{1,5} & a_{5,3} & a_{5,4} - \lambda a_{1,4}a_{1,5} & a_{5,5} \end{pmatrix}.$$

Proof. As a first step, we derive combinatorial expressions for $\det A$, $\det B$ and $\det C$.

Let us introduce some notation. As earlier, we write $\mathfrak{S}_n$ for the symmetric group on $\{1, 2, \dots, n\}$. Given a permutation $\sigma \in \mathfrak{S}_n$, we let $c(\sigma)$ be the number of cycles of σ. Furthermore we define the function f_1 by

$$f_1(\sigma) = \begin{cases} 1 & \text{if } \sigma(1) = 1, \\ 0 & \text{if } \sigma(1) \neq 1. \end{cases}$$

By the definition of the determinant, we have

$$\det A = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma \prod_{i=1}^{n} A_{i,\sigma(i)} = \sum_{\sigma\in\mathfrak{S}_n} (-1)^{n-c(\sigma)}\lambda^{1-f_1(\sigma)} \prod_{i=1}^{n} a_{i,\sigma(i)}. \tag{4}$$

In order to describe combinatorial expressions for $\det B$ and $\det C$, we need some further notations and definitions. We write $\tilde{\mathfrak{S}}_{n-1}$ for the symmetric group on $\{2, 3, \dots, n\}$. A *signed* permutation on $\{2, 3, \dots, n\}$ is a pair (π, ε), where $\pi \in \tilde{\mathfrak{S}}_{n-1}$ and $\varepsilon \in \{-1, 1\}^{n-1}$. For the sake of convenience, we label the components of ε from 2 through n, that is, $\varepsilon = (\varepsilon_2, \varepsilon_3, \dots, \varepsilon_n)$. We define the *weight* $w(\pi, \varepsilon)$ of a signed permutation (π, ε) by

$$w(\pi, \varepsilon) = \prod_{i=2}^{n} W_{i,\pi(i)},$$

where

$$W_{i,\pi(i)} = \begin{cases} a_{i,\pi(i)} & \text{if } \varepsilon_i = 1, \\ -\lambda a_{1,i}a_{1,\pi(i)} & \text{if } \varepsilon_i = -1. \end{cases}$$

We need two particular subsets of all signed permutations: let $\mathrm{SP}^{(1)}_{n-1}$ denote the set of all signed permutations on $\{2,3,\dots,n\}$ with $\varepsilon_i=1$ whenever $i+\pi(i)$ is odd, $2\le i\le n$, and let $\mathrm{SP}^{(2)}_{n-1}$ denote the set of all signed permutations on $\{2,3,\dots,n\}$ with $\varepsilon_i=1$ whenever $i+\pi(i)$ is even, $2\le i\le n$.

Now, with all this notation, for the determinant of B we have

$$\det B=\sum_{\sigma\in\tilde{\mathfrak{S}}_{n-1}}\operatorname{sgn}\sigma\prod_{i=2}^{n}B_{i,\sigma(i)}=\sum_{(\pi,\varepsilon)\in\mathrm{SP}^{(1)}_{n-1}}(\operatorname{sgn}\pi)\cdot w(\pi,\varepsilon),\tag{5}$$

while for the determinant of C we have

$$\det C=\sum_{\sigma\in\tilde{\mathfrak{S}}_{n-1}}\operatorname{sgn}\sigma\prod_{i=2}^{n}B_{i,\sigma(i)}=\sum_{(\pi,\varepsilon)\in\mathrm{SP}^{(2)}_{n-1}}(\operatorname{sgn}\pi)\cdot w(\pi,\varepsilon).\tag{6}$$

We can now begin the actual proof of (3).

We start by identifying the coefficients of λ^0: By inspection, the coefficient of λ^0 in $\det A$ is $a_{1,1}\det(a_{i,j})_{2\le i,j\le n}$. It is equally obvious that the coefficient of λ^0 in $\det B$, as well as in $\det C$, is equal to $\det(a_{i,j})_{2\le i,j\le n}$. Thus, the coefficients of λ^0 on both sides of (3) agree.

Next we identify the coefficients of λ^1: clearly, the coefficient of λ^1 on the right-hand side of (3) is zero. We now use expressions (4)–(6) for $\det A$, $\det B$, and $\det C$, respectively, to show that this is also the case on the left-hand side. The coefficient of λ^1 in (4) is

$$\begin{aligned}\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(1)\ne1}}(-1)^{n-c(\sigma)}\prod_{i=1}^{n}a_{i,\sigma(i)}&=\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(1)\ne1}}(-1)^{n-c(\sigma)}a_{\sigma^{-1}(1),1}a_{1,\sigma(1)}\prod_{\substack{i=2\\ i\ne\sigma^{-1}(1)}}^{n}a_{i,\sigma(i)}\\&=\beta\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(1)\ne1}}(-1)^{n-c(\sigma)}a_{1,\sigma^{-1}(1)}a_{1,\sigma(1)}\prod_{\substack{i=2\\ i\ne\sigma^{-1}(1)}}^{n}a_{i,\sigma(i)}.\end{aligned}\tag{7}$$

Terms contributing to the coefficient of λ^1 in (5) and (6) occur exactly for the signed permutations (π,ε) where ε is a vector with exactly one component equal to -1. Let $\varepsilon^{(k)}$ denote the vector with all components equal to 1 except for the k-th, which is -1. The coefficient of λ^1 in (5) is then

$$\sum_{\pi\in\tilde{\mathfrak{S}}_{n-1}}\sum_{\substack{k=2\\ k+\pi(k)\text{ even}}}^{n}(\operatorname{sgn}\pi)\cdot\frac{w(\pi,\varepsilon^{(k)})}{\lambda}=-\sum_{\pi\in\tilde{\mathfrak{S}}_{n-1}}\sum_{\substack{k=2\\ k+\pi(k)\text{ even}}}^{n}(\operatorname{sgn}\pi)\cdot a_{1,k}a_{1,\pi(k)}\prod_{\substack{i=2\\ i\ne k}}^{n}a_{i,\pi(i)},$$

while the coefficient of λ^1 in (6) is

$$\sum_{\pi\in\tilde{\mathfrak{S}}_{n-1}}\sum_{\substack{k=2\\ k+\pi(k)\text{ odd}}}^{n}(\operatorname{sgn}\pi)\cdot\frac{w(\pi,\varepsilon^{(k)})}{\lambda}=-\sum_{\pi\in\tilde{\mathfrak{S}}_{n-1}}\sum_{\substack{k=2\\ k+\pi(k)\text{ odd}}}^{n}(\operatorname{sgn}\pi)\cdot a_{1,k}a_{1,\pi(k)}\prod_{\substack{i=2\\ i\ne k}}^{n}a_{i,\pi(i)}.\tag{8}$$

The coefficient of λ^1 in the sum of (5) and (6) is therefore

$$-\sum_{\pi\in\tilde{\mathfrak{S}}_{n-1}}\sum_{k=2}^{n}(-1)^{n-1-c(\pi)}\cdot a_{1,k}a_{1,\pi(k)}\prod_{\substack{i=2\\i\neq k}}^{n}a_{i,\pi(i)}. \tag{9}$$

To a fixed pair (π,k), $\pi\in\tilde{\mathfrak{S}}_{n-1}$ and $2\le k\le n$, we now associate the permutation $\sigma\in\mathfrak{S}_n$, given by $\sigma(k)=1$, $\sigma(1)=\pi(k)$, and $\sigma(i)=\pi(i)$ for all $i\neq 1,k$. Thus we see that (7) and (9) multiplied by β are identical, which implies that the coefficient of λ^1 on the left-hand side of (3) vanishes, as desired.

The remaining task is to show that all other terms in the sum of (5) and (6) cancel. The reader should note that these "other terms" in (5) and (6) are indexed by signed permutations (π,ε) in the union $\mathrm{SP}^{(1)}_{n-1}\cup\mathrm{SP}^{(2)}_{n-1}$, where the vector ε has at *least two* components equal to -1. We show that these terms cancel by defining a *sign-reversing involution* i on the set of signed permutations in $\mathrm{SP}^{(1)}_{n-1}\cup\mathrm{SP}^{(2)}_{n-1}$, where the vector ε has at least two components equal to -1. The map i will be defined separately on three disjoint subsets of this set.

Set 1. Consider all signed permutations (π,ε) in $\mathrm{SP}^{(1)}_{n-1}\cup\mathrm{SP}^{(2)}_{n-1}$ with the property that there are at least two *even* indices i_1,i_2, $2\le i_1<i_2\le n$, with $\varepsilon_{i_1}=\varepsilon_{i_2}=-1$. Let us call this property the property P1.

Given a signed permutation (π,ε) with property P1, let i_1 and i_2 be even integers such that $\varepsilon_{i_1}=\varepsilon_{i_2}=-1$, which are minimal with respect to this property. Then we define

$$i\big((\pi,\varepsilon)\big):=(\pi\circ(i_1,i_2),\varepsilon). \tag{10}$$

(The permutation $\pi\circ(i_1,i_2)$ is the composition of π and the transposition exchanging i_1 and i_2.) Clearly, $i\big((\pi,\varepsilon)\big)$ also has property P1 since the vector ε has not changed. Furthermore, if $(\pi,\varepsilon)\in\mathrm{SP}^{(1)}_{n-1}$, then also $i\big((\pi,\varepsilon)\big)\in\mathrm{SP}^{(1)}_{n-1}$, and similarly for $\mathrm{SP}^{(2)}_{n-1}$. By definition, the weight of (π,ε) is

$$w(\pi,\varepsilon)=(-\lambda a_{1,i_1}a_{1,\pi(i_1)})(-\lambda a_{1,i_2}a_{1,\pi(i_2)})\prod_{\substack{i=2\\i\neq i_1,i_2}}^{n}W_{i,\pi(i)},$$

while the weight of $i\big((\pi,\varepsilon)\big)$ is

$$w\big(\pi\circ(i_1,i_2),\varepsilon\big)=(-\lambda a_{1,i_1}a_{1,\pi(i_2)})(-\lambda a_{1,i_2}a_{1,\pi(i_1)})\prod_{\substack{i=2\\i\neq i_1,i_2}}^{n}W_{i,\pi(i)},$$

which is equal to $w(\pi,\varepsilon)$. In summary, we have established the relation

$$(\operatorname{sgn}\pi)\cdot w(\pi,\varepsilon)=-(\operatorname{sgn}(\pi\circ(i_1,i_2)))\cdot w\big(\pi\circ(i_1,i_2),\varepsilon\big).$$

Since, in addition, i is an involution, the terms in (5) indexed by signed permutations with property P1 cancel each other pairwise, and the same is true for the analogous terms in (6).

Set 2. Now we consider all signed permutations in $\mathrm{SP}_{n-1}^{(1)} \cup \mathrm{SP}_{n-1}^{(2)}$ which do not have property P1, but have the property that there are at least two *odd* indices i_1, i_2, $2 \le i_1 < i_2 \le n$ with $\varepsilon_{i_1} = \varepsilon_{i_2} = -1$. Let us call this property the property P2. Given a signed permutation (π, ε) with property P2, we define the map i by (10), as before. It is then easy to see that everything else is as in the previous case. In particular, the terms in (5) indexed by signed permutations with property P2 cancel each other pairwise, and the same is true for the analogous terms in (6).

Set 3. Finally, we consider the signed permutations (π, ε) in $\mathrm{SP}_{n-1}^{(1)} \cup \mathrm{SP}_{n-1}^{(2)}$ which have neither property P1 nor property P2. Since ε must have at least two components equal to -1, the only possibility is then that there is an *even* i_1 and an *odd* i_2 such that $\varepsilon_{i_1} = \varepsilon_{i_2} = -1$, and these are the only components of ε which are equal to -1. We define the map i again by (10). This time, if $(\pi, \varepsilon) \in \mathrm{SP}_{n-1}^{(1)}$, then $i\big((\pi, \varepsilon)\big) \in \mathrm{SP}_{n-1}^{(2)}$, and vice versa. However, all the other conclusions of the first case remain valid, and, thus, again the terms in the sum of the right-hand sides of (5) and (6) indexed by signed permutations in this subset cancel each other pairwise.

This completes the proof of (3). $\square$

Corollary 4. *Let $(a_{i,j})_{1 \le i,j \le n}$ be the doubly indexed sequence with the property that $a_{i,j} = a_{j,i}$ and $a_{i,i} = 2$ for all i and j. Then, if the matrices A, B and C are defined as in Theorem 3 (with $\beta = 1$), we have*

$$\det A = \det B + \det C.$$

Proof. We set $\beta = 1$, $a_{i,j} = a_{j,i}$ and $a_{i,i} = 2$ for all i and j in Theorem 3. Then $a_{1,1} - 2\beta = 0$, and, hence, the assertion is equivalent to Equation (3) with these specializations. $\square$

Corollary 5. *Let $(a_{i,j})_{1 \le i,j \le n}$ be the doubly indexed sequence with the property that $a_{i,j} = -a_{j,i}$ for all i and j. Then, if the matrices A, B and C are defined as in Theorem 3 (with $\beta = -1$), for even n we have*

$$\det A + \det B + \det C = 0.$$

Proof. We set $\beta = -1$ and $a_{i,j} = -a_{j,i}$ for all i and j in Theorem 3. Then the determinant on the right-hand side of (3) is the determinant of a skew-symmetric matrix of odd size and, hence, zero. The assertion is therefore equivalent to Equation (3) with these specializations. $\square$

As it turns out, in the "skew-symmetric" case (that is, in the case of Corollary 5), the determinant of the matrix C factors into two big factors. These two factors can be described explicitly. They are the "even" and the "odd" parts of the Pfaffian of the skew-symmetric matrix $(a_{i,j})_{1\le i,j\le n}$. Recall that, by definition, for even n the Pfaffian of a skew-symmetric matrix $A=(a_{i,j})_{1\le i,j\le n}$, denoted $\operatorname{Pf}(A)$, is the square root of $\det A$, where the sign is chosen so that the term $a_{1,n}a_{2,n-1}\cdots a_{n/2,n/2+1}$ occurs with coefficient $+1$. In combinatorial terms, the Pfaffian of A is (cf. [9, Sec. 2])

$$\operatorname{Pf}(A)=\sum_m \operatorname{sgn} m\cdot\prod_{(i,j)\in m} a_{i,j}, \tag{11}$$

where the sum is over all perfect matchings m on $\{1,2,\dots,n\}$. Let $\operatorname{Pf}_e(A)$ denote the sum of all the terms appearing on the right-hand side of (11) which contain $a_{1,k}$ for an *even* k. Similarly, we denote by $\operatorname{Pf}_o(A)$ the sum of all the terms appearing in $\operatorname{Pf}(A)$ which contain $a_{1,k}$ for an *odd* k.

Theorem 6. *Let n be even, and let $A=(a_{i,j})_{1\le i,j\le n}$ be a skew-symmetric matrix, that is $a_{i,j}=-a_{j,i}$ for all i and j. If C is defined as in Theorem 3 (with $\beta=-1$), that is,*

$$C_{i,j}=\begin{cases} a_{i,j} & \text{if } i+j \text{ is even,}\\ a_{i,j}-\lambda a_{1,i}a_{1,j} & \text{if } i+j \text{ is odd,}\end{cases}$$

then we have

$$\det C=-2\lambda\operatorname{Pf}_e(A)\operatorname{Pf}_o(A).$$

Proof. We expand the determinant $\det(C)$ as in the proof of Theorem 3, see (6). Clearly, in this expansion, all the contributions of signed permutations in $\mathrm{SP}^{(2)}_{n-1}$ for which all the ε_i's are 1 cancel each other, because the sum of these contributions is simply $\det(a_{i,j})_{2\le i,j\le n}$, the determinant of a skew-symmetric matrix of odd dimension. The arguments given in the proof of Theorem 3 for Sets 1 and 2 show in addition that the contributions of signed permutations in $\mathrm{SP}^{(2)}_{n-1}$ for which there exist i_1 and i_2, both even or both odd, such that $\varepsilon_{i_1}=\varepsilon_{i_2}=-1$ cancel each other. However, this happens also for signed permutations in Set 3, i.e., for signed permutations in $\mathrm{SP}^{(2)}_{n-1}$ for which $\varepsilon_{i_1}=\varepsilon_{i_2}=-1$ for an even i_1 and an odd i_2, and for which all other ε_i's are 1. To see this, for fixed even i_1 and odd i_2, let us consider all the signed permutations (π,ε) in $\mathrm{SP}^{(2)}_{n-1}$ with $\varepsilon_{i_1}=\varepsilon_{i_2}=-1$, $\varepsilon_i=1$ otherwise, with a fixed value $\pi(i_1)$ and a fixed value $\pi(i_2)$. Their contribution to (6) is

$$\lambda^2\operatorname{sgn}(i_1,i_2,\pi(i_1),\pi(i_2))\cdot a_{1,i_1}a_{1,\pi(i_1)}a_{1,i_2}a_{1,\pi(i_2)}\det A^{1,i_1,i_2}_{1,\pi(i_1),\pi(i_2)}, \tag{12}$$

where $A^{1,i_1,i_2}_{1,\pi(i_1),\pi(i_2)}$ is the submatrix of A which arises by deleting the rows numbered 1, i_1 and i_2 and the columns numbered 1, $\pi(i_1)$ and $\pi(i_2)$, and where $\operatorname{sgn}(i_1,i_2,\pi(i_1),\pi(i_2))$ is a certain sign. If we interchange the roles of i_1 and $\pi(i_1)$, and of i_2 and $\pi(i_2)$, and consider the analogous signed permutations in $\mathrm{SP}^{(2)}_{n-1}$, then their contribution is

$$\lambda^2\operatorname{sgn}(i_1,i_2,\pi(i_1),\pi(i_2))\cdot a_{1,i_1}a_{1,\pi(i_1)}a_{1,i_2}a_{1,\pi(i_2)}\det A^{1,\pi(i_1),\pi(i_2)}_{1,i_1,i_2}. \tag{13}$$

However, we have

$$A^{1,i_1,i_2}_{1,\pi(i_1),\pi(i_2)} = -\left(A^{1,\pi(i_1),\pi(i_2)}_{1,i_1,i_2}\right)^t,$$

which implies

$$\det A^{1,i_1,i_2}_{1,\pi(i_1),\pi(i_2)} = -\det A^{1,\pi(i_1),\pi(i_2)}_{1,i_1,i_2},$$

since these are determinants of matrices of dimension $n-3$, which is odd. Thus, the sum of the terms (12) and (13) is zero.

In summary, the above arguments have shown that $\det C$ is equal to the contributions in (6) by signed permutations in $\mathrm{SP}^{(2)}_{n-1}$ with *exactly* one ε_i which is -1. To be precise, they show that (compare with expression (8))

$$\det C = -\lambda \sum_{\pi\in\tilde{\mathfrak{S}}_{n-1}} \sum_{\substack{k=2\\ k+\pi(k)\text{ odd}}}^{n} (\operatorname{sgn}\pi)\cdot a_{1,k}a_{1,\pi(k)}\prod_{\substack{i=2\\ i\neq k}}^{n} a_{i,\pi(i)}.$$

In fact, there is still cancellation in the expression on the right-hand side. If $\pi\in\tilde{\mathfrak{S}}_{n-1}$ should have a cycle of odd length which does not contain k, then the permutation $\bar\pi$ arising from π by reversing the orientation of the cycle has the same sign, but the product $\prod_{i=2,i\neq k}^{n} a_{i,\bar\pi(i)}$ has sign opposite to $\prod_{i=2,i\neq k}^{n} a_{i,\pi(i)}$. Thus, the contributions corresponding to π and $\bar\pi$ cancel each other. An analogous argument shows that the same is true if the cycle of π containing k should have an even length. Thus,

$$\det C = -\lambda \sum{}' \sum_{\substack{k=2\\ k+\pi(k)\text{ odd}}}^{n} (\operatorname{sgn}\pi)\cdot a_{1,k}a_{1,\pi(k)}\prod_{\substack{i=2\\ i\neq k}}^{n} a_{i,\pi(i)}, \tag{14}$$

where the sum $\sum{}'$ is over all $\pi\in\tilde{\mathfrak{S}}_{n-1}$ with only cycles of even length except that the cycle containing k has odd length.

To show that the expression (14) is equal to $-2\lambda\,\mathrm{Pf}_e(A)\,\mathrm{Pf}_o(A)$, we construct a bijection between $\tilde{\mathfrak{S}}_{n-1}$ and $M_e\times M_o\times\{1,-1\}$, where M_e denotes the set of all perfect matchings on $\{1,2,\dots,n\}$ with the property that 1 is matched to an even number, and where M_o is the analogous set of perfect matchings with the property that 1 is matched to an odd number. If π is mapped to (m_1,m_2,η) under this bijection, then this bijection will have the property that

$$\operatorname{sgn}\pi\cdot a_{1,k}a_{1,\pi(k)}\prod_{\substack{i=2\\ i\neq k}}^{n} a_{i,\pi(i)} = -(\operatorname{sgn} m_1)(\operatorname{sgn} m_2)\prod_{(i,j)\in m_1} a_{i,j}\prod_{(i,j)\in m_2} a_{i,j}.$$

Clearly, given such a bijection, the assertion of the theorem would be proved.

Let $\pi\in\tilde{\mathfrak{S}}_{n-1}$. Consider a cycle of π not containing k. Let i be the smallest number in the cycle. Then we match i to $\pi(i)$ in m_1, $\pi(i)$ to $\pi^2(i)$ in m_2, $\pi^2(i)$ to $\pi^3(i)$ in m_1, $\pi^3(i)$ to $\pi^4(i)$ in m_2, etc. Considering the cycle containing k, we let $\eta=1$ if k is even while we let $\eta=-1$ is k is odd. If k is even, then we match 1 to k in m_1, k to

$\pi(k)$ in m_2, $\pi(k)$ to $\pi^2(k)$ in m_1, etc., while if k is odd, we match 1 to k in m_2, k to $\pi(k)$ in m_1, $\pi(k)$ to $\pi^2(k)$ in m_2, etc. It is obvious that this sets up a bijection. The fact that the sign behaves in the correct way under the bijection can be shown in a similar manner as in the proof of Proposition 2.2 in [9]. $\square$

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