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CONSISTENT FAMILIES OF MEASURES AND THEIR EXTENSIONS

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Introduction

For any n "one-dimensional" random variables $\xi_1, \cdots, \xi_n$ with arbitrary sets of values, there exists (see, for example, Halmos, [1], § 37) an "n-dimensional" random variable ξ, such that the variables $\xi_1, \cdots, \xi_n$ are its projections. Such a variable may be formed, for instance, if it is assumed that the projections $\xi_1, \cdots, \xi_n$ are independent. However, it is not always possible to construct a random variable with given consistent projections. For example, if

$$P(\xi_1 = 1 \cap \xi_2 = 1) = P(\xi_1 = 0 \cap \xi_2 = 0) = \tfrac{1}{2}, \tag{1}$$

$$P(\xi_1 = 1 \cap \xi_3 = 1) = P(\xi_1 = 0 \cap \xi_3 = 0) = \tfrac{1}{2}, \tag{2}$$

$$P(\xi_2 = 1 \cap \xi_3 = 0) = P(\xi_2 = 0 \cap \xi_3 = 1) = \tfrac{1}{2}, \tag{3}$$

then the probability $P(\xi_1 = 0 \cap \xi_2 = 0 \cap \xi_3 = 0)$ cannot be assigned a non-negative value, consistent with (1)—(3).

Thus if we take three two-dimensional (in the usual sense) random variables (X_1, Y_1), (X_2, Z_2) and (Y_3, Z_3), for which the distribution functions of X_1 and X_2, Y_1 and Y_3, and Z_2 and Z_3, respectively, coincide, then it may not be possible to find a three-dimensional random variable (X, Y, Z), whose two-dimensional projections have distributions which coincide with the distribution functions of the two-dimensional random variables (X_1, Y_1), (X_2, Z_2), and (Y_3, Z_3). Some necessary conditions for the existence of such a random variable were pointed out by Bass [2]. Schell established a connection between certain economic problems and the problem of finding a three-dimensional random variable with given consistent two-dimensional projections.

It will be shown that the existence of a random variable with given higher-dimensional projections is closely connected with the combinatorial properties of the "complex of coordinates" of these projections, the skeletons of which are sets of coordinates, belonging to the various projections. Thus, in the example given above, the complex is the boundary of a triangle. The "cyclicity" of this complex is, in the last analysis, the reason why there sometimes does not exist a random variable with given two-dimensional consistent projections.

In application to the theory of games the foregoing means that a combination of the players into groups can be found, so that the mixed coordinated actions of these groups cannot determine any mixed manner of action for all the players of the game which is consistent with the actions of these groups. These types of questions, developed in line with the requirements of the theory of coalition games, can find future applications also in the theory of information and the theory of random algorithms.

The present paper establishes necessary and sufficient conditions which must be imposed on the complex of coordinates in order that random variables with any prescribed pairwise consistent projections exist. This fact was communicated by the author without proof in the note [4].

1. Regular Skeleton Complexes

1.1. In what follows a complex will mean a finite unrestricted skeleton complex, in the combinatorial-topological sense of this term (see, for example, [5]).

The set of all vertices of the complex $\mathfrak{K}$ we shall sometimes denote by $|\mathfrak{K}|$. If $\mathfrak{L}$ is some set of skeletons of $\mathfrak{K}$ (not necessarily unrestricted), then by $|\mathfrak{L}|$ we shall understand the set of all vertices of $\mathfrak{K}$ which belong to at least one skeleton of $\mathfrak{L}$.

1.2. We shall encounter some complexes particularly often, so that it is convenient to assign them special notations.

The complex consisting of all subsets of some set of n elements (an $(n-1)$-dimensional abstract closed simplex) will be denoted by I_n. The complex consisting of all subsets of some concrete set R will be denoted by I_R.

The complex consisting of all proper subsets of a set of n elements (the boundary of the $(n-1)$-dimensional closed simplex) we shall denote by G_n. The complex consisting of skeletons of the form $\{a_1, a_2\}$, $\{a_2, a_3\}, \cdots, \{a_{n-1}, a_n\}$, $\{a_n, a_1\}$ and the corresponding vertices (the one-dimensional cycle of n-links), we denote by Z_n.

1.3. A skeleton T of a complex $\mathfrak{K}$ will be called a *maximal* skeleton, if it is not a proper face of some other skeleton of $\mathfrak{K}$.

The number of maximal skeletons of the complex $\mathfrak{K}$ is called its *length*, and denoted by $\kappa(\mathfrak{K})$. It is clear that for $\kappa(\mathfrak{K}) = 1$, it is necessary and sufficient that $\mathfrak{K} = I_n$.

1.4. Let T_1 and T_2 be two different maximal skeletons of $\mathfrak{K}$. We shall say that T_1 yields a maximal intersection with T_2 if the intersection $T_1 \cap T_2$ is not a proper face of the intersection $T_1 \cap T_3$ for some skeleton $T_3 \in \mathfrak{K}$.

A maximal skeleton $T \in \mathfrak{K}$ is called *extreme* in $\mathfrak{K}$, if all maximal intersections of T with skeletons of $\mathfrak{K}$ are equal.

We stipulate that the empty complex does not have any extreme skeletons.

If T is an extreme skeleton of the complex $\mathfrak{K}$, then there is a vertex $a \in T$ which does not belong to any maximal skeleton of $\mathfrak{K}$, different from T (such vertices of an extreme skeleton will be called its *proper vertices*). Indeed, let R be the maximal intersection of T with skeletons of $\mathfrak{K}$. $T-R \neq \Lambda$, since otherwise the skeleton T would not be maximal. It is clear that any vertex of $T-R$ is a proper vertex of T.

The set of all proper vertices of an extreme skeleton T of the complex $\mathfrak{K}$ we denote by $\sigma_{\mathfrak{K}} T$.

1.41. Let T be an extreme skeleton in the complex $\mathfrak{K}$. Consider the subcomplex of $\mathfrak{K}$, consisting of all skeletons of $\mathfrak{K}$ which do not intersect $\sigma_{\mathfrak{K}} T$. This subcomplex we call the *normal subcomplex* of $\mathfrak{K}$, *corresponding to the extreme skeleton* T.

A subcomplex $\tilde{\mathfrak{K}}$ of the complex $\mathfrak{K}$ is called *a normal subcomplex,* if it is the normal subcomplex, corresponding to some extreme skeleton of $\mathfrak{K}$.

Any sequence of subcomplexes of the complex $\mathfrak{K}$:

$$\mathfrak{K} = \mathfrak{K}_0 \supset \mathfrak{K}_1 \supset \cdots \supset \mathfrak{K}_r, \tag{4}$$

in which for any $l = 1, \cdots, r-1$, the complex $\mathfrak{K}_{l+1}$ is a normal subcomplex of the complex $\mathfrak{K}_l$, and the complex $\mathfrak{K}_r$ does not have any extreme skeletons (and therefore also no normal subcomplexes) we call a *normal series* of the complex $\mathfrak{K}$. The number r is called the *length* of the normal series of the complex $\mathfrak{K}$.

It is not difficult to show (in a manner similar to the proof of the Jordan-Hölder theorem; see, for example, [6]) that, for any complex, all of its normal series have the same length.

1.5. The complex $\mathfrak{K}$ is called *regular,* if it possesses a normal series in which the last term is empty.

The class of all regular complexes will be denoted by $\mathfrak{R}$.

It is not difficult to see that for any regular complex the length of its normal series is equal to $\kappa(\mathfrak{K})$. This property is characteristic of regular complexes, as can be shown without difficulty; the length of the normal series of any non-regular complex $\mathfrak{K}$ will be strictly less than $\kappa(\mathfrak{K})$.

We note that the complex I_n is regular for any n, and G_n and Z_n are not regular for $n \geqq 3$.

We note further that the regularity of a complex is not a topological invariant and cannot be, in general, expressed in terms of its homological properties, since, for example, a triangulation of a regular complex may or may not be regular. Thus if a two-dimensional simplex is triangulated by joining any interior point with all vertices, a non-regular complex is clearly obtained.

1.51. Let us establish some properties of regular complexes.

Lemma. *Any regular complex $\mathfrak{K}$ for which $\kappa(\mathfrak{K}) \geqq 2$ has at least two extreme skeletons.*

The proof is carried out by induction on $\kappa(\mathfrak{K})$. If $\kappa(\mathfrak{K}) = 2$, then, clearly, each of the maximal skeletons of $\mathfrak{K}$ is extreme, and the assertion is proved.

We assume that the lemma holds for all complexes with n maximal skeletons, and that $\kappa(\mathfrak{K}) = n+1$.

Let T be some extreme skeleton of $\mathfrak{K}$, T^* an arbitrary fixed maximal skeleton of $\mathfrak{K}$, having a maximal intersection with T, and $\tilde{\mathfrak{K}}$ the normal subcomplex of $\mathfrak{K}$, corresponding to the extreme skeleton T.

Let T_1 be any extreme skeleton of $\tilde{\mathfrak{K}}$, which is not extreme in $\mathfrak{K}$. This means that in $\mathfrak{K}$ there are two different maximal skeletons T' and T'', which have different maximal intersections with T_1. Of these skeletons only one may belong to $\tilde{\mathfrak{K}}$ (otherwise the skeleton T_1 would not be extreme in $\tilde{\mathfrak{K}}$). The other skeleton must therefore be equal to T (for convenience we take $T'' = T$). But $T \cap T_1 = T \cap T^*$ by the maximality of the latter intersection with T, and therefore

(5) $$T \cap T_1 \subset T^* \cap T_1,$$

and in view of the maximality of the intersection $T \cap T_1 = T'' \cap T_1$ in T_1

(6) $$T \cap T_1 \supset T^* \cap T_1.$$

From relations (5) and (6) we obtain that $T \cap T_1 = T^* \cap T_1$.

If now $T^* \neq T_1$, then since the intersection $T \cap T_1$ is maximal in T_1, the intersection $T^* \cap T_1$ is also maximal in T_1, so that T_1 has different maximal intersections with skeletons of the complex $\tilde{\mathfrak{K}}$ (namely, with the skeletons T' and T^*). This, however, cannot occur, since the skeleton T_1 is extreme in the complex $\mathfrak{K}$. Consequently $T^* = T_1$. Since this equality holds for any extreme skeleton T_1 of the complex $\tilde{\mathfrak{K}}$, which is not extreme in $\mathfrak{K}$, the equality of all such complexes follows.

Thus, in the transition from the complex $\mathfrak{K}$ to the complex $\tilde{\mathfrak{K}}$ precisely one extreme skeleton is lost (namely T) and at most one (perhaps T_1) is gained. Therefore the transition from $\mathfrak{K}$ to $\tilde{\mathfrak{K}}$ does not increase the number of extreme skeletons, so if there are not less than two in $\tilde{\mathfrak{K}}$, there must be at least two in $\mathfrak{K}$.

The lemma is proved.

1.52. Lemma. *Let $\mathfrak{K}$ be a cone over $\tilde{\mathfrak{K}}$ with vertex a. Then the complexes $\mathfrak{K}$ and $\tilde{\mathfrak{K}}$ are either both regular or both irregular.*

The proof follows immediately from the fact that every skeleton $T \in \tilde{\mathfrak{K}}$ corresponds to the skeleton $T \cup a \in \mathfrak{K}$, and under this correspondence maximal skeletons correspond to maximal ones, and the intersection of skeletons to their intersections.

1.53. Lemma. *If the complex $\mathfrak{K}$ is regular, and $a \in |\mathfrak{K}|$, then the complex $\tilde{\mathfrak{K}} = \mathfrak{K} - O_{\mathfrak{K}} a$ is also regular.*

Proof. Let $\mathfrak{K} = \mathfrak{K}_0 \supset \mathfrak{K}_1 \supset \cdots \supset \mathfrak{K}_n = \Lambda$ be a normal series of $\mathfrak{K}$, where T_i is the extreme skeleton of $\mathfrak{K}_i$, $i = 0, \cdots, n-1$, which corresponds to the normal subcomplex $\mathfrak{K}_{i+1}$. We note first of all the obvious fact that

$$\bigcup_{i=0}^{n} \sigma_{\mathfrak{K}_i} T_i = |\mathfrak{K}|$$

and we fix here an i_0, for which

(7) $$a \in \sigma_{\mathfrak{K}_{i_0}} T_{i_0}.$$

Let $\tilde{\mathfrak{K}}_i = \mathfrak{K}_i - O_{\mathfrak{K}} a$. Let, for the time being,

(8) $$i \neq i_0.$$

Denote by T a skeleton of $\mathfrak{K}_{i+1}$, with which the extreme skeleton T_i has maximal intersection.

If now $a \notin T_i$, then *a fortiori* $a \notin T_i \cap T$, and in addition $T_i \in \tilde{\mathfrak{K}}_i$. Therefore the intersection $T_i \cap (T-a)$ contains all intersections of T_i with the remaining skeletons of $\tilde{\mathfrak{K}}_i$, so that the skeleton T_i is extreme in the complex $\tilde{\mathfrak{K}}_i$. It is clear that the normal subcomplex of $\tilde{\mathfrak{K}}_i$, corresponding to T_i, is $\tilde{\mathfrak{K}}_{i+1}$.

If, on the other hand, $a \in T_i$, then, in view of (7) and (8), $a \in T \cap T_i$. This means that $T-a \in \tilde{\mathfrak{K}}_i$. All intersections of T_i-a with skeletons of $\tilde{\mathfrak{K}}_i$ are contained in the intersection $(T_i-a) \cap (T-a)$, and the skeleton T_i-a is extreme in the complex $\tilde{\mathfrak{K}}_i$. It is furthermore clear that

$$\begin{aligned}\tilde{\mathfrak{K}}_{i+1} &= \mathfrak{K}_{i+1}-O_{\mathfrak{K}}a = \mathfrak{K}_i-\Big(\bigcup_{b \in \sigma_{\mathfrak{K}_i}T_i} O_{\mathfrak{K}_i}b \cup O_{\mathfrak{K}}a\Big)\\ &= (\mathfrak{K}_i-O_{\mathfrak{K}}a)-\bigcup_{b \in \sigma_{\mathfrak{K}_i}T_i} O_{\mathfrak{K}_i}b = \tilde{\mathfrak{K}}_i-\bigcup_{b \in \sigma_{\tilde{\mathfrak{K}}_i}(T_i-a)} O_{\tilde{\mathfrak{K}}_i}b.\end{aligned}$$

We now turn to the case $i = i_0$. Here it is easy to see that T_i-a is an extreme skeleton of $\tilde{\mathfrak{K}}_{i_0}$, and $\tilde{\mathfrak{K}}_{i_0+1} = \mathfrak{K}_{i_0+1}$ and is a normal subcomplex of $\tilde{\mathfrak{K}}_{i_0}$, with the exception of the case when the set $\sigma_{\mathfrak{K}_{i_0}}T_{i_0}$ consists of the single element a. Then $\tilde{\mathfrak{K}}_{i_0+1} = \mathfrak{K}_{i_0}$.

Thus the sequence $\tilde{\mathfrak{K}} = \tilde{\mathfrak{K}}_0 \supset \cdots \supset \tilde{\mathfrak{K}}_n = \Lambda$ is a normal series of $\tilde{\mathfrak{K}}$, if $\sigma_{\mathfrak{K}_{i_0}}T_{i_0} \neq a$. In the opposite case, this sequence is turned into a normal series, if the term $\tilde{\mathfrak{K}}_{i_0}$ is dropped from it. The complex $\tilde{\mathfrak{K}}$ is thus shown to be regular.

1.531. Corollary. *If the complex $\mathfrak{K}$ is regular, and $A \subset |\mathfrak{K}|$, then the complex $\mathfrak{K}-\bigcup_{a \in A} O_{\mathfrak{K}}a$ is also regular.*

This follows from the preceding, and from the obvious fact that

$$\bigcup_{a \in A} O_{\mathfrak{K}}a = \bigcup_{i=0}^{k-1} O_{\mathfrak{K}_i}a_{i+1}, \tag{9}$$

where $\{a_1, \cdots, a_k\} = A$, $\mathfrak{K}_0 = \mathfrak{K}$, and $\mathfrak{K}_{i+1} = \mathfrak{K}_i-O_{\mathfrak{K}_i}a_{i+1}$, $i = 0, \cdots, k-1$.

1.54. Sometimes the lemma just proved admits a converse.

Lemma. *Let $a \in |\mathfrak{K}|$, $|O_{\mathfrak{K}}a|$ be a skeleton of $\mathfrak{K}$, and the complex $\tilde{\mathfrak{K}} = \mathfrak{K}-O_{\mathfrak{K}}a$ be regular. Then the complex $\mathfrak{K}$ is also regular.*

Proof. We form a normal series of the complex $\tilde{\mathfrak{K}}$:

$$\tilde{\mathfrak{K}} = \tilde{\mathfrak{K}}_0 \supset \cdots \supset \tilde{\mathfrak{K}}_r = \Lambda.$$

Let $\tilde{\mathfrak{K}}_i$ be the last of the subcomplexes occurring in this series which contains the skeleton $T-a$. Let

$$\mathfrak{K}_e = \begin{cases} \tilde{\mathfrak{K}}_e \cup I_{T-a}, & e \leqq i,\\ \tilde{\mathfrak{K}}_e, & e > i.\end{cases}$$

It is clear that if $e \neq i$, then the complex $\mathfrak{K}_{e+1}$ is a normal subcomplex of the complex $\mathfrak{K}_e$. This will also take place in the case $e = i$, if the skeleton $T-a$ is maximal in $\tilde{\mathfrak{K}}_i$ (because then it must be extreme in $\tilde{\mathfrak{K}}_i$ so that the skeleton T will be extreme in $\mathfrak{K}_i$). If $T-a$ is not maximal in $\tilde{\mathfrak{K}}_i$, i.e. is a face of some (necessarily extreme!) skeleton T^*, then the skeleton T will be extreme in $\mathfrak{K}_i$, and $\tilde{\mathfrak{K}}_i$ will be a normal subcomplex of $\mathfrak{K}_i$.

In the first of these cases a normal series of the complex will be the sequence $\mathfrak{K}_0 \supset \mathfrak{K}_1 \supset \cdots \supset \mathfrak{K}_r$, and in the second, the sequence $\mathfrak{K}_0 \supset \mathfrak{K}_1 \supset \cdots \supset \mathfrak{K}_i \supset \tilde{\mathfrak{K}}_i \supset \mathfrak{K}_{i+1} \supset \cdots \supset \mathfrak{K}_r$.

2. An Abstract Characterization of the Class of all Regular Complexes

2.1. In the future we shall have to deal not only with individual regular complexes, but also with the entire class of these complexes, and therefore it is necessary to give an abstract characterization of this class. It is not difficult to show that the class $\mathfrak{R}$ of all regular complexes can be so characterized.

Let $\mathfrak{T}$ be the class of complexes satisfying the following conditions:

1°. $\mathfrak{T}$ contains all complexes of the form I_n, $n \geqq 1$.
2°. If $\mathfrak{K} \in \mathfrak{T}$, $T \in \mathfrak{K}$, $R \cap |\mathfrak{K}| = \Lambda$, then $\mathfrak{K} \cup I_{T \cup R} \in \mathfrak{T}$.
3°. $\mathfrak{T}$ is the smallest class of complexes which satisfies conditions 1° and 2°.

Then $\mathfrak{T} = \mathfrak{R}$.

The characterization just given of the class $\mathfrak{R}$, for all its combinatorial naturality, suffers from the defect that condition 3° is not sufficiently constructive. Therefore for our purposes it will be necessary to supplement this characterization with another, which is at first glance somewhat artificial.

2.2. Theorem. *Let $\mathfrak{T}$ be the class of complexes satisfying the following properties*:

1°. $\mathfrak{R} \subset \mathfrak{T}$.
2°. *If $\mathfrak{K} \in \mathfrak{T}$ and $a \in |\mathfrak{K}|$, then*

$$\mathfrak{K} - O_{\mathfrak{K}}a \in \mathfrak{T}. \tag{10}$$

3°. *If $a \in |\mathfrak{K}|$, $\mathfrak{K} - O_{\mathfrak{K}}a \in \mathfrak{T}$, and*

$$|O_{\mathfrak{K}}a| \in \mathfrak{K}, \tag{11}$$

then $\mathfrak{K} \in \mathfrak{T}$.

4°. *For $n \geqq 3$, $G_n \notin \mathfrak{T}$.*
5°. *For $n \geqq 3$, $Z_n \notin \mathfrak{T}$.*

Then $\mathfrak{T} = \mathfrak{R}$.

The proof of this theorem will be carried out by contradiction. Suppose that $\mathfrak{R} \neq \mathfrak{T}$. By condition 1° this means that there exist complexes in $\mathfrak{T} - \mathfrak{R}$. Let $\mathfrak{K}$ be minimal among these (i.e. such that any subcomplex of $\mathfrak{K}$ either belongs to $\mathfrak{R}$ or does not belong to $\mathfrak{T}$).

We consider first the case when the complex $\mathfrak{K}$ possesses a vertex a, such that

$$|O_{\mathfrak{K}}a| = |\mathfrak{K}|. \tag{12}$$

This means that, for any vertex $b \in |\mathfrak{K}|$

(13) $$\{a, b\} \in \mathfrak{K}.$$

If here $T \bigcup a \in \mathfrak{K}$ for every $T \notin O_{\mathfrak{K}} a$, then $\mathfrak{K}$ is a cone over $\mathfrak{K}-O_{\mathfrak{K}} a$ with vertex a. But by the assumption of minimality for $\mathfrak{K}$, either

(14) $$\mathfrak{K}-O_{\mathfrak{K}} a \in \mathfrak{N},$$

or $\mathfrak{K}-O_{\mathfrak{K}} a \notin \mathfrak{T}$. In the first of these cases, by Lemma 1.52, $\mathfrak{K} \in \mathfrak{N}$. In the second case, by condition 2°, $\mathfrak{K} \notin \mathfrak{T}$. The desired contradiction has been found.

Let there now exist skeletons $T \notin O_{\mathfrak{K}} a$, such that

(15) $$T \bigcup a \notin \mathfrak{K}.$$

These skeletons have not less than two vertices, since otherwise (13) would contradict (15). Let k be the smallest number of vertices of a skeleton of this type ($k \geqq 2$) and let T have precisely k vertices. For every proper face T' of T (since T' has less than k vertices) $T' \bigcup a \in \mathfrak{K}$. Since, in addition, $T \in \mathfrak{K}$, $\mathfrak{K}$ contains all proper faces of the skeleton $T \bigcup a$, and does not contain this skeleton itself. Deleting from $\mathfrak{K}$ the stars of all of its vertices which are not in $T \bigcup a$, we obtain G_{k+1}, $k \geqq 2$. But $G_{k+1} \notin \mathfrak{T}$ by 4°. Therefore on the basis of 2° and formula (9) $\mathfrak{K} \notin \mathfrak{T}$. The case (12) thus is solved.

Thus we may henceforth assume that for any vertex a of the complex $\mathfrak{K}$, $|\mathfrak{K}|-|O_{\mathfrak{K}} a| \neq \Lambda$. We may further assume that for any vertex $a \in |\mathfrak{K}|$,

(16) $$|O_{\mathfrak{K}} a| \notin \mathfrak{K},$$

since otherwise (11) would be fulfilled, and by 2° and 3° the complex $\mathfrak{K}$ belongs to $\mathfrak{T}$ if and only if the complex $\mathfrak{K}-O_{\mathfrak{K}} a$ belongs to $\mathfrak{T}$. In the first of these cases by the minimality of $\mathfrak{K}$, (14) would have to be fulfilled, and therefore by Lemma 1.54, $\mathfrak{K} \in \mathfrak{N}$. In the second case, however, $\mathfrak{K} \notin \mathfrak{T}$.

We shall therefore assume that (16) holds for any vertex a. Let $a_1 \in |\mathfrak{K}|$. By (16), $[O_{\mathfrak{K}} a_1]$[1] differs from complexes of the form I_n and therefore contains more than one maximal skeleton. Further, $R = |\mathfrak{K}|-|O_{\mathfrak{K}} a_1| \neq \Lambda$. Therefore

(17) $$[O_{\mathfrak{K}} a_1] = \mathfrak{K}- \bigcup_{r \in R} O_{\mathfrak{K}} r \neq \mathfrak{K}.$$

By 2° it follows from $\mathfrak{K} \in \mathfrak{T}$ that $[O_{\mathfrak{K}} a_1] \in \mathfrak{T}$ and, by the minimality of $\mathfrak{K}$, from (17) it follows that $[O_{\mathfrak{K}} a_1] \in \mathfrak{K}$.

From what has been proved it follows that the complex $[O_{\mathfrak{K}} a_1]$ has extreme skeletons. Let T_1 be one of these and a_2 its proper vertex in the complex $[O_{\mathfrak{K}} a_1]$. In view of the fact that a_1 is a vertex of *every* maximal skeleton of $[O_{\mathfrak{K}} a_1]$ (and there are more than one of these in the case under consideration), a_1 cannot be a proper vertex of T_1, and therefore, in particular, $a_2 \neq a_1$.

Consider the star $O_{\mathfrak{K}} a_2$. Since $[O_{\mathfrak{K}} a_2] \neq \mathfrak{K}$, this complex, like the complex $[O_{\mathfrak{K}} a_1]$, is regular. Since (16) remains valid for it, by the lemma of 1.51, this complex has at least two extreme skeletons. Let T_2 be an extreme skeleton of $[O_{\mathfrak{K}} a_2]$, different from T_1. Let a_3 be a proper vertex of T_2 in $[O_{\mathfrak{K}} a_2]$. As before, we see that $a_3 \neq a_2$. Further, T_1, $T_2 \in [O_{\mathfrak{K}} a_2]$. Therefore, if $a_3 = a_1$, then $a_3 \in T_1$, so that the vertex a_3 would not be a *proper* vertex of T_2 in $[O_{\mathfrak{K}} a_2]$.

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[1] As is obvious, $[O_{\mathfrak{K}} a]$ denotes $O_{\mathfrak{K}} a \cup B_{\mathfrak{K}} a$, i.e. the closed star of the vertex a in $\mathfrak{K}$.

The set $\{a_1, a_2, a_3\}$ is not a skeleton of $\mathfrak{K}$. Indeed, if it were, then $\{a_1, a_2, a_3\} \in O_{\mathfrak{K}} a_1$. But a_1 is a proper vertex of T_1 in $O_{\mathfrak{K}} a_1$. Since it is also a vertex of any skeleton of $[O_{\mathfrak{K}} a_1]$, such a skeleton, in the present case $\{a_1, a_2, a_3\}$, must be a face of T_1. But then the vertex a_3 must be a vertex of T_1, and we have already seen that it is not.

Considering the closure of the star $O_{\mathfrak{K}} a_3$, we find in it an extreme skeleton T_3, different from T_2, and choose in it a proper vertex a_4. Analogous to the above, we see that $a_4 \neq a_2$, $a_4 \neq a_3$, and the set $\{a_2, a_3, a_4\}$ is not a skeleton of $\mathfrak{K}$. Continuing this process, we obtain a sequence

$$a_1, a_2, a_3, \cdots \tag{18}$$

of vertices of $\mathfrak{K}$, which we now consider.

The pairs $\{a_1, a_2\}$, $\{a_2, a_3\}$, $\cdots$ of vertices of the above sequence are skeletons of $\mathfrak{K}$. It is clear that there exist (for example, by the finiteness of the set of vertices of the complex $\mathfrak{K}$) r and s such that

$$\{a_r, a_{r+1}\}, \{a_{r+1}, a_{r+2}\}, \cdots, \{a_{s-1}, a_s\}, \{a_s, a_r\} \in \mathfrak{K}. \tag{19}$$

We take the shortest sequence of two-element skeletons of this type. Let the sequence (19) be one of these. Then it is clear that if $r \leqq i, j \leqq s$, then $\{a_i, a_j\} \in \mathfrak{K}$ only in the case when $|i-j| = 1$, or $|i-j| = s-r$.

By the construction of the sequence (18), $s-r \geqq 3$. Let $\mathfrak{K}^*$ be the subcomplex of $\mathfrak{K}$, consisting of the skeletons (19) and their vertices. It is clear that $\mathfrak{K}^*$ has the form

$$\mathfrak{K} - \bigcup_{a \in |\mathfrak{K}| - A} O_{\mathfrak{K}} a, \qquad A = \{a_r, \cdots, a_s\}, \tag{20}$$

and also $\mathfrak{K}^* = Z_{s-r}$, $s-r \geqq 3$.

By 5°, $\mathfrak{K}^* \notin \mathfrak{T}$, so that by 2° and (20), we must have also $\mathfrak{K} \notin \mathfrak{T}$.

The theorem is proved.

3. Generalized Measurable Spaces and Consistent Families of Measures

3.1. A set S, considered along with some system Σ of σ-algebras of its subsets, will be called a *generalized measurable space* and will be denoted by $\langle S, \Sigma \rangle$.

Let $\langle S, \mathfrak{S}_1 \rangle$ and $\langle S, \mathfrak{S}_2 \rangle$ be two measurable spaces, and μ_1 and μ_2 be measures corresponding to them. These measures will be called *consistent*, if $\mu_1(A) = \mu_2(A)$ for any $A \in \mathfrak{S}_1 \cap \mathfrak{S}_2$.

Let $\langle S, \Sigma \rangle$ be a generalized measurable space; moreover, set to each σ-algebra $\mathfrak{S} \in \Sigma$ of subsets of S a measure $\mu_{\mathfrak{S}}$ in correspondence.

The generalized measurable space $\langle S, \Sigma \rangle$, considered along with the family of measures μ_Σ, consisting of all measures $\mu_{\mathfrak{S}}$, $\mathfrak{S} \in \Sigma$, will be called a *generalized space with measures*. A generalized space with measures will be denoted by the triplet $\langle S, \Sigma, \mu_\Sigma \rangle$.

The family of measures μ_Σ will be called a *consistent* family of measures on $\langle S, \Sigma \rangle$, if the measures occurring in it are pairwise consistent.

It is clear that in the particular case when Σ consists of a single σ-algebra $\mathfrak{S}$, the generalized measurable space turns out to be the usual measurable space.

In this case the family of measures μ_Σ consists of one measure, so that the generalized space with measures $\langle S, \Sigma, \mu_\Sigma\rangle$ is just the usual space with measure, and the family μ_Σ itself is trivially consistent.

3.11. Let $\langle S, \Sigma\rangle$ be a generalized measurable space, let μ'_Σ and μ''_Σ be two families of measures on it, and let α_1 and α_2 be arbitrary real numbers. For every $\mathfrak{S} \in \Sigma$ and $A \in \mathfrak{S}$ we set

$$(\alpha'\mu'_{\mathfrak{S}}+\alpha''\mu''_{\mathfrak{S}})(A) = \alpha'\mu'_{\mathfrak{S}}(A)+\alpha''\mu''_{\mathfrak{S}}(A).$$

It is clear that if the numbers α' and α'' are non-negative, the functions $\alpha'\mu'_{\mathfrak{S}}+\alpha''\mu''_{\mathfrak{S}}$ are measures on $\langle S, \mathfrak{S}\rangle$, so that we may speak of the family of measures $\alpha'\mu'_\Sigma+\alpha''\mu''_\Sigma$ on the generalized measurable space $\langle S, \Sigma\rangle$.

If here the families of measures μ'_Σ and μ''_Σ are consistent, then the family $\alpha'\mu'_\Sigma+\alpha''\mu''_\Sigma$ is also consistent.

3.2. Let $[\Sigma]$ be the smallest σ-algebra of subsets of S which contains $\bigcup_{\mathfrak{S}\in\Sigma}\mathfrak{S}$. The measures μ on the measurable space $\langle S, [\Sigma]\rangle$ is called an *extension* of the family of measures μ_Σ on the generalized measurable space $\langle S, \Sigma\rangle$, if it is consistent with every measure of μ_Σ. If such a measure exists, then the family of measures is called *extendable.*

In the same way the generalized space with measures $\langle S, \Sigma, \mu_\Sigma\rangle$ will be called *extendable,* if there exists a space with measure $\langle S, [\Sigma], \mu\rangle$, so that the measure μ is an extension of the family μ_Σ. In this case $\langle S, [\Sigma], \mu\rangle$ is also called an extension of $\langle S, \Sigma, \mu_\Sigma\rangle$.

The question of the extendability of consistent families of measures on generalized measurable spaces is extremely complex. On one hand, examples of non-extendable consistent families can be found even among the simplest structures of this kind. One such example was indicated in the introduction. On the other hand, there exist rather wide classes of spaces with measures, for which such extensions are possible. We turn to one of those.

3.3. Let M be some set. To each $m \in M$ we set in correspondence a measurable space $\langle S_m, \mathfrak{S}^0_m\rangle$ and consider for some $K \subset M$ the Cartesian product $S_K = \prod_{m\in K} S_m$.

Let $\mathfrak{P}_K$ be the set of all measurable rectangular K-cylinders in S_M, i.e. the family of all subsets having the form

$$\prod_{m\in K} A_m \times S_{M-K}, \qquad A_m \in \mathfrak{S}^0_m.$$

By $\mathfrak{S}_K$ we shall understand the smallest σ-algebra of subsets of S_M which contains $\mathfrak{P}_K$.

Thus to the subsets $K \subset M$ are associated the measurable spaces $\langle S_M, \mathfrak{S}_K\rangle$.

3.31. It is clear that from $K_1 \subset K_2$ follows $\mathfrak{P}_1 \subset \mathfrak{P}_2$, and thus $\mathfrak{S}_1 \subset \mathfrak{S}_2$. This means that for the consistency of the measures μ_{K_1} and μ_{K_2} on the measurable spaces $\langle S_M, \mathfrak{S}_{K_1}\rangle$ and $\langle S_M, \mathfrak{S}_{K_2}\rangle$, it suffices to establish that $\mu_{K_1}(A) = \mu_{K_2}(A)$ for any $A \in \mathfrak{S}_{K_1}$.

3.32. Let $\mathfrak{K}$ be an arbitrary family of subsets of M. The generalized measurable space, consisting of the set S_M and the σ-algebras of the form $\mathfrak{S}_K$,

$K \in \mathfrak{K}$, on it, formed according to the preceding paragraph, we denote by $\langle S_M, \Sigma_{\mathfrak{K}}\rangle$.

3.33. Let M be a finite set, and $\mathfrak{K}$ a complex of its subsets, where $|\mathfrak{K}| = M$ and $\mu_{\mathfrak{K}}$ is a consistent family of measures on the generalized measurable space $\langle S_M, \Sigma_{\mathfrak{K}}\rangle$.

It is clear that for the measure μ on $\langle S_M, \mathfrak{S}_M\rangle$ to be an extension of the family $\mu_{\mathfrak{K}}$, it suffices that this measure be consistent with every measure of the form μ_T, where T is a maximal skeleton of the complex $\mathfrak{K}$.

4. The Regularity of Complexes and the Extendability of Consistent Families of Measures

4.1. Let again M be a finite set, and $\mathfrak{K}$ some complex of its subsets, and let $\langle S_M, \Sigma_{\mathfrak{K}}, \mu_{\mathfrak{K}}\rangle$ be a generalized measurable space with consistent measures. In the future the set S_M will always be assumed finite. Moreover, without loss of generality, we can assume that every σ-algebra $\mathfrak{S}_a^0$, $a \in |\mathfrak{K}|$, consists of all subsets of S_a.

Set $x_K^* = x_K \times S_{M-K}$ for each $K \subset M$, and $x_K \in S_K$. It is clear that the sets x_K^* are minimal subsets of the σ-algebra $\mathfrak{S}_K$. If $x \in S_M$, then x_K is the projection of x on S_K.

4.2. The question of the extendability of a consistent family of measures $\mu_{\mathfrak{K}}$ on the generalized measurable space $\langle S_M, \Sigma_{\mathfrak{K}}\rangle$ is closely related to the regularity of the complex $\mathfrak{K}$. Indeed, we have the following theorem.

Theorem. *Let $\mathfrak{K}$ be some complex of subsets of the set M. In order that any consistent family of measures $\mu_{\mathfrak{K}}$ on any (finite) generalized measurable space $\langle S_M, \Sigma_{\mathfrak{K}}\rangle$ be extendable, it is necessary and sufficient that the complex $\mathfrak{K}$ be regular.*

PROOF. Let $\mathfrak{T}$ be the class of all those complexes $\mathfrak{K}$, for which any consistent family of measures $\mu_{\mathfrak{K}}$ on any space $\langle S_M, \Sigma_{\mathfrak{K}}\rangle$ is extendable. We shall show that this class $\mathfrak{T}$ satisfies the conditions $1°-5°$ of Theorem 2.2.

$1°$. Let $\mathfrak{K} \in \mathfrak{R}$. The proof that $\mathfrak{K} \in \mathfrak{T}$ is carried out by induction on $\kappa(\mathfrak{K})$.

If $\kappa(\mathfrak{K}) = 1$, then $\mathfrak{K} = I_n$, $|\mathfrak{K}| \in \mathfrak{K}$, and the family $\mu_{\mathfrak{K}}$ already contains its extension $\mu_{|\mathfrak{K}|}$.

Assume that we have already shown that the class $\mathfrak{T}$ contains all regular complexes $\mathfrak{K}$ for which $\kappa(\mathfrak{K}) \leqq n$. Consider a regular complex $\mathfrak{K}$ with $\kappa(\mathfrak{K}) = n+1$. Let T be one of its extreme skeletons, and let $\tilde{\mathfrak{K}}$ be the normal subcomplex of $\mathfrak{K}$, corresponding to T. It is clear that $\tilde{\mathfrak{K}} \in \mathfrak{R}$ and $\kappa(\tilde{\mathfrak{K}}) = n$. Therefore $\tilde{\mathfrak{K}} \in \mathfrak{T}$.

We construct the measure $\mu_{|\mathfrak{K}|}$ on S_M in the following way. Denote $|\tilde{\mathfrak{K}}| \cap T$ by V and set for any $x \in S_{|\mathfrak{K}|}$ for which $\mu_V(x_V^*) \neq 0$

$$\mu_{|\mathfrak{K}|}(X) = \frac{\mu_{|\tilde{\mathfrak{K}}|}(x^*_{|\tilde{\mathfrak{K}}|})\mu_T(x_T^*)}{\mu_V(x_V^*)}, \tag{21}$$

and when $\mu_V(x_V^*) = 0$, set $\mu_{|\mathfrak{K}|}(x) = 0$. For any $X \subset S_{|\mathfrak{K}|}$, we shall set $\mu_{|\mathfrak{K}|}(X)$

$= \sum_{x \in X} \mu_{|\mathfrak{K}|}(x)$. It is clear that $\mu_{|\mathfrak{K}|}$ is a measure on $S_{|\mathfrak{K}|}$. We shall show that it is consistent with the measures μ_T and $\mu_{|\tilde{\mathfrak{K}}|}$.

Indeed, let $\bar{X} \in \mathfrak{P}_{|\tilde{\mathfrak{K}}|}$. It is clear that then $\bar{X} = A \times B \times S_{T-V}$, where $A \subset S_{|\tilde{\mathfrak{K}}|-V}$ and $B \subset S_V$. Therefore, by (21),

$$(22) \quad \mu_{|\mathfrak{K}|}(\bar{X}) = \sum_{\substack{a \in A,\, b \in B \\ x \in S_{T-V}}} \mu_{|\mathfrak{K}|}(a \times b \times x) = \sum_{\substack{a \in A,\, b \in B' \\ x \in S_{T-V}}} \frac{\mu_{|\tilde{\mathfrak{K}}|}((b \times x)^*)\mu_T((a \times b)^*)}{\mu_V(b^*)},$$

where B' is the set of all $b \in B$ such that $\mu_V(b^*) \neq 0$. Further

$$\mu_{|\mathfrak{K}|}(\bar{X}) = \sum_{a \in A,\, b \in B'} \frac{\mu_T((a \times b)^*) \sum_{x \in S_{T-V}} \mu_{|\tilde{\mathfrak{K}}|}((b \times x)^*)}{\mu_V(b^*)} = \sum_{a \in A,\, b \in B'} \frac{\mu_T((a \times b)^*)\mu_{|\tilde{\mathfrak{K}}|}(b^*)}{\mu_V(b^*)}.$$

But $\mu_{|\mathfrak{K}|}(b^*) = \mu_V(b^*)$, in view of the assumed consistency of the measures $\mu_{|\tilde{\mathfrak{K}}|}$ and μ_V, and since $b \in B'$, we obtain from (22)

$$(23) \qquad \mu_{|\mathfrak{K}|}(\bar{X}) = \sum_{a \in A,\, b \in B'} \mu_T((a \times b)^*).$$

Finally, from the consistency of the measures μ_V and μ_T it follows that when $\mu_V(b^*) = 0$

$$\mu_T((a \times b)^*) \leqq \mu_T(b^*) = 0.$$

Therefore (23) can be written as

$$\mu_{|\mathfrak{K}|}(\bar{X}) = \sum_{\substack{a \in A \\ b \in B'}} \mu_T((a \times b)^*) = \mu_T(\bar{X}),$$

and, in view of paragraph **3.31.** the consistency of the measures $\mu_{|\mathfrak{K}|}$ and μ_T is proved.

In an analogous manner we prove the consistency of the measures $\mu_{|\mathfrak{K}|}$ and $\mu_{|\tilde{\mathfrak{K}}|}$, and, in considering the complex $I_{|\tilde{\mathfrak{K}}|} \cup I_T$, we now need only refer to the assertions of **3.33.**

2°. Let $\mathfrak{K} \in \mathfrak{T}$, $a \in |\mathfrak{K}|$, $\tilde{\mathfrak{K}} = \mathfrak{K} - O_{\mathfrak{K}} a$, and $\mu_{\tilde{\mathfrak{K}}}$ be an arbitrary consistent family of measures on the generalized measurable space $\langle S_{|\tilde{\mathfrak{K}}|}, \Sigma_{\tilde{\mathfrak{K}}} \rangle$. We shall show that the family $\mu_{\tilde{\mathfrak{K}}}$ is extendable.

Consider the measurable space $\langle S_a, \mathfrak{S}_a^0 \rangle$, in which the set S_a consists only of the single element α, and set, for any $T \in \tilde{\mathfrak{K}}$,

$$(24) \quad \tilde{\mathfrak{S}}_{T \cup a} = \tilde{\mathfrak{S}}_T = \{X \times S_a\}_{X \in \mathfrak{S}_T}, \quad \tilde{\mu}_{T \cup a}(X \times S_a) = \tilde{\mu}_T(X \times S_a) = \mu_T(X).$$

Setting $\{\tilde{\mathfrak{S}}_T\}_{T \in \mathfrak{K}} = \tilde{\Sigma}_{\mathfrak{K}}$ and $\{\tilde{\mu}_T\}_{T \in \tilde{\mathfrak{K}}} = \tilde{\mu}_{\mathfrak{K}}$, we obtain a generalized space with measures $\langle S_{|\mathfrak{K}|}, \tilde{\Sigma}_{\mathfrak{K}}, \tilde{\mu}_{\mathfrak{K}} \rangle$.

The family of measures $\tilde{\mu}_{\mathfrak{K}}$ is clearly consistent and therefore (since $\mathfrak{K} \in \mathfrak{T}$) is extendable. Let $\tilde{\mu}_{|\mathfrak{K}|}$ be one of its extensions. We form the measure $\mu_{|\tilde{\mathfrak{K}}|}$ by setting $\mu_{|\tilde{\mathfrak{K}}|}(X) = \tilde{\mu}_{|\mathfrak{K}|}(X \times S_a)$ for any $T \in \tilde{\mathfrak{K}}$ and $X \in \mathfrak{S}_T$. In this case, since the measures $\tilde{\mu}_{|\mathfrak{K}|}$ and $\tilde{\mu}_T$ are consistent and because of (24),

$$\mu_{|\tilde{\mathfrak{K}}|}(X) = \tilde{\mu}_T(X \times S_a) = \mu_T(X),$$

which means that the measure $\mu_{|\tilde{\mathfrak{K}}|}$ is consistent with each of the measures μ_T, i.e., the family $\mu_{\tilde{\mathfrak{K}}}$ is extendable.

3°. Let $a \in |\mathfrak{K}|$, $|O_{\mathfrak{K}} a| \in \mathfrak{K}$, $\tilde{\mathfrak{K}} = \mathfrak{K} - O_{\mathfrak{K}} a \in \mathfrak{T}$ and $\mu_{\mathfrak{K}}$ be a consistent family of measures on $\langle S_{|\mathfrak{K}|}, \Sigma_{\mathfrak{K}} \rangle$. Let further $T \in \tilde{\mathfrak{K}}$ and $X \in \mathfrak{S}_T$. Then $X = \tilde{X} \times S_a$, where $\tilde{X} \in \mathfrak{S}_{|\tilde{\mathfrak{K}}|}$. Set $\tilde{\mathfrak{S}}_T = \{\tilde{X} : \tilde{X} \times S_a \in \mathfrak{S}_T\}$ and for any $\tilde{X} \in \tilde{\mathfrak{S}}_T$

$$\tilde{\mu}_T(\tilde{X}) = \mu_T(\tilde{X} \times \mathfrak{S}_a).$$

In the end we obtain a family of measures $\tilde{\mu}_{\tilde{\mathfrak{K}}} = \{\tilde{\mu}_T\}_{T \in \tilde{\mathfrak{K}}}$ on the generalized measurable space $\langle S_{|\tilde{\mathfrak{K}}|}, \{\tilde{\mathfrak{S}}_T\}_{T \in \tilde{\mathfrak{K}}} \rangle$. This family is consistent and (since $\tilde{\mathfrak{K}} \in \mathfrak{T}$) extendable. Let $\tilde{\mu}_{|\tilde{\mathfrak{K}}|}$ be its extension.

We form for any $L \subset |\tilde{\mathfrak{K}}|$ the algebra of subsets $\tilde{\mathfrak{S}}_L$, generated by the rectangular subsets $\tilde{\mathfrak{P}}_L \subset S_{|\tilde{\mathfrak{K}}|}$, and we set $\tilde{\mu}_L(X) = \tilde{\mu}_{|\tilde{\mathfrak{K}}|}(X)$ for any $X \in \tilde{\mathfrak{S}}_L$. Finally, we take $\mu_L(X \times S_a) = \tilde{\mu}_L(X)$.

The measures μ_L are defined on the algebras of subsets $\mathfrak{S}_L$, and it is clear that they are consistent. Along with the measures μ_T, $T \in \mathfrak{K} - \tilde{\mathfrak{K}}$, they form a consistent family of measures $\mu_{\mathfrak{L}}$ on the generalized measurable space $\langle S_{|\mathfrak{K}|}, \Sigma_{\mathfrak{L}} \rangle$, where $\mathfrak{L}$ denotes the complex consisting of all subsets of $|\tilde{\mathfrak{K}}|$ and all subsets of $|O_{\mathfrak{K}} a|$. The complex $\mathfrak{L}$ is clearly regular $(\kappa(\mathfrak{L}) = 2)$, so that by point 1° of the proof of the theorem the family of measures $\mu_{\mathfrak{L}}$ is extendable, which is what was to be proved.

4°. Let $\mathfrak{K} = G_n$, $n \geqq 3$. We shall show that $\mathfrak{K} \notin \mathfrak{T}$. Let $a_1, \cdots, a_n$ be the vertices of the complex $\mathfrak{K}$. If a_i is some vertex of $\mathfrak{K}$, then $\mathfrak{K}_i$ will denote the complex $\mathfrak{K} - O_{\mathfrak{K}} a_i$.

We form, in correspondence with each vertex a_i of the complex $\mathfrak{K}$, the measurable space $\langle S_i, \mathfrak{S}_i^0 \rangle$, where the set S_i consists of two elements, α_i and β_i, and $\mathfrak{S}_i^0$ of all subsets of S_i, and we consider the generalized measurable space $\langle S_{|\mathfrak{K}|}, \Sigma_{\mathfrak{K}} \rangle$.

Let ξ be an n-termed sequence of zeros and ones:

$$\xi = (\xi_1, \cdots, \xi_n). \tag{25}$$

Set

$$\xi^{(i)} = (\xi_1, \cdots, \xi_{i-1}, \xi_{i+1}, \cdots, \xi_n). \tag{26}$$

The sequences of the form ξ, and also of the form $\xi^{(i)}$ for any fixed i, will be assumed to be ordered lexicographically. For $i < j$ the sequence $\xi^{(i)}$ will be assumed to precede any sequence $\xi^{(j)}$.

Each element $\gamma \in S_{|\mathfrak{K}|}$ has the form $\gamma_1 \times \cdots \times \gamma_n$, where $\gamma_i = \alpha_i, \beta_i$. To this element we set a sequence $\xi(\gamma)$ of the form (25) in correspondence, where $\xi_i = 0$ or 1, depending on whether $\gamma_i = \alpha_i$ or $\gamma_i = \beta_i$.

Analogously each element $\gamma \in S_{|\mathfrak{K}_i|}$ has the form $\gamma_1 \times \cdots \times \gamma_{i-1} \times \gamma_{i+1} \times \cdots \times \gamma_n$, where $\gamma_j = \alpha_j, \beta_j, j \neq i$. To such an element γ we let correspond the sequence $\xi^{(i)}(\gamma)$ of the form (26), where $\xi_j^{(i)}(\gamma) = 0$ or 1, if $\gamma_j = \alpha_j$ or β_j, respectively.

The order of the elements of $S_{|\mathfrak{K}|}$, and also of the elements of the sets $S_{|\mathfrak{K}_i|}$ will be taken corresponding to the order of the sequences of the form (25) and (26) which correspond to them.

We put in correspondence with each measure $\mu_{|\mathfrak{K}|}$ on $S_{|\mathfrak{K}|}$ the point in the 2^n-dimensional vector space E, the coordinates of which are the numbers

$\mu_{|\mathfrak{R}|}(\gamma)$, $\gamma \in S_{|\mathfrak{R}|}$, written in the order which we have given to the elements of the set $S_{|\mathfrak{R}|}$. It is clear that then the unit basis vectors of the space E will correspond to degenerate measures on $S_{|\mathfrak{R}|}$, and the set of all points which correspond to measures on $S_{|\mathfrak{R}|}$ will be the (2^n-1)-dimensional simplex spanned by these basis vectors.

Further, to every consistent family of measures $\mu_{\mathfrak{R}}$ on $\langle S_{|\mathfrak{R}|}, \Sigma_{\mathfrak{R}}\rangle$ we let correspond the point in the $n2^{n-1}$-dimensional vector space E', the coordinates of which are the numbers

$$\mu_{|\mathfrak{R}_i|}(\gamma), \qquad i = 1, \cdots, n; \gamma = \gamma_i^{(i)} \times S_i, \gamma^{(i)} \in S_{|\mathfrak{R}_i|}, \tag{27}$$

written in the order prescribed for the elements of the sets $S_{|\mathfrak{R}_i|}$. The set of all points of E obtained in this way will be denoted by R.

It is clear that among the points of R will be all those which correspond to extendable families of measures. We denote the set of these by R^*. But a family of measures $\mu_{\mathfrak{R}}$ is extendable if and only if all measures of $\mu_{\mathfrak{R}}$ are induced by some measure $\mu_{|\mathfrak{R}|}$. Moreover,

$$\mu_{|\mathfrak{R}_i|}(\gamma \times S_i) = \mu_{|\mathfrak{R}|}(\gamma \times \alpha_i) + \mu_{|\mathfrak{R}|}(\gamma \times \beta_i) \tag{28}$$

for any $i = 1, \cdots, n$ and $\gamma \in S_{|\mathfrak{R}_i|}$.

Formulas (28) determine a linear mapping of the space E into the space E'. Let A be the matrix of this mapping. It is easily seen that under our ordering of the basis vectors of E the matrix A has the form

$$\begin{Vmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{Vmatrix}.$$

It is easily seen that the rank of this matrix is 2^n-1. The unique linear relation among its columns is

$$\sum_{j \in \mathbf{E}} A_{\cdot j} = \sum_{j \in \mathbf{O}} A_{\cdot j}, \tag{29}$$

where $\mathbf{E}(\mathbf{O})$ denotes the set of those numbers among $0, 1, \cdots, 2^n-1$ whose binary representation has an even (respectively, and odd) number of ones, and $A_{\cdot j}$ denotes, as usual, the j-th column of the matrix A. We emphasize that the extreme left column of the matrix A is the zeroth column.

We note two obvious but important properties of this matrix.

First, of the two non-zero elements of each row of the matrix, one lies in a column with a number from **E**, and the other in a column with a number from **O**. Second, the ones in the first column lie in the first row, and also in those rows whose numbers have the form $k2^{n-1}+2$, $k = 1, \cdots, n-1$. The second one of each of these rows lies in the zeroth column, or the column with number 2^k+1, respectively.

The columns of A are the images of the basis vectors of E after their transformation by the matrix A. Consequently, the set R^*, being the image of the (2^n-1)-dimensional simplex spanned by these basis vectors, is some (2^n-2)-dimensional convex polyhedron.

We shall show that R^* has at least one boundary (in the relative sense) point, all the coordinates of which are positive.

Let

$$0 < \lambda < \lambda' \tag{30}$$

be arbitrary numbers for which

$$n\lambda' + (2^{n-1}-n)\lambda = 1. \tag{31}$$

Let $\mathbf{E}^*$ be the set $\mathbf{E}$, with the number 0 and all numbers of the form $1+2^k$, $k = 1, \cdots, n-1$, deleted, and form the vector

$$V(\alpha) = (1+\alpha)\left(\lambda'\left(A_{\cdot 0}+\sum_{k=1}^{n-1} A_{\cdot 1+2^k}\right)+\lambda \sum_{j\in \mathbf{E}^*} A_{\cdot j}\right)-\alpha A_{\cdot 1} \tag{32}$$

depending on the parameter α.

By (31), the vector

$$V(0) = \lambda'\left(A_{\cdot 0}+\sum_{k=1}^{n-1} A_{\cdot 1+2^k}\right)+\lambda \sum_{j\in \mathbf{E}^*} A_{\cdot j}$$

is a linear convex combination of the columns of the matrix A, i.e. the vertices of R^*, and therefore $V(0) \in R^*$. All components of $V(0)$ are clearly positive. If $V(0)$ belongs to the boundary of R^*, then the sought for point is found.

Let now $V(0)$ be a (relatively) interior point of R^*. We consider the vector $V[\lambda'/(1-\lambda')]$. Its first component, and also each of its components with numbers $k2^{n-1}+2$, $k = 1, \cdots, n-1$, are equal to zero by virtue of the second of the above remarks on properties of the matrix A:

$$\left(1+\frac{\lambda'}{1-\lambda'}\right)\lambda' - \frac{\lambda'}{1-\lambda'} = 0.$$

The remaining components of this vector are positive, by the first property of A. Therefore each of the components of any vector $V(\alpha)$, $0 \leqq \alpha < \lambda'/(1-\lambda')$, are positive.

We shall show that $V[\lambda'/(1-\lambda')] \notin R^*$. The convex set R^* is a polyhedron, i.e. has a finite number of extreme points. Therefore (see, for example [7]) R^* is the convex hull of its vertices. Consequently, if $V[\lambda'/(1-\lambda')] \in R^*$, then we could find non-negative $\rho_0, \rho_1, \cdots, \rho_{2^n-1}$, such that

$$\sum_{k=0}^{2^n-1} \rho_k = 1 \quad \text{and} \quad V\left(\frac{\lambda'}{1-\lambda'}\right) = \sum_{k=0}^{2^n-1} \rho_k A_{\cdot k}.$$

But then we would, by (32), have

$$(33)\qquad \left(1+\frac{\lambda'}{1-\lambda'}\right)\left(\lambda'\left(A_{\cdot 0}+\sum_{k=1}^{n-1} A_{\cdot 1+2^k}\right)+\lambda \sum_{k \in \mathbf{E}^*} A_{\cdot k}\right)-\frac{\lambda'}{1-\lambda'}=\sum_{k=0}^{2^n-1} \rho_k A_{\cdot k}.$$

Since, however, there is no relation between the columns of the matrix A, except (29), relation (33) must be identical to (29), i.e., after bringing all terms to one side all the coefficients of $A_{\cdot k}$, $k \in \mathbf{E}$, must be equal to one another, and the coefficients of $A_{\cdot k}$, $k \in \mathbf{O}$, must differ from them only by sign. But these coefficients equal

$$(34)\qquad \frac{\lambda'}{1-\lambda'}-\rho_k \quad \text{for} \quad A_{\cdot k} \quad \text{if} \quad k=0 \quad \text{or} \quad 1+2^t, \quad t=1,\cdots,n-1,$$

$$(35)\qquad -\frac{\lambda'}{1-\lambda'}-\rho_1 \quad \text{for} \quad A_{\cdot 1},$$

$$(36)\qquad \left(1+\frac{\lambda'}{1-\lambda'}\right)\lambda-\rho_k \quad \text{for} \quad A_{\cdot k} \quad \text{if} \quad k \in \mathbf{E}^*,$$

$$(37)\qquad -\rho_k \quad \text{for} \quad A_{\cdot k} \quad \text{if} \quad k \in \mathbf{O}.$$

Comparing (34) and (35), we see that $\rho_k+\rho_1=0$, $k=0, 1+2^t$, $t=1,\cdots, n-1$, and since these terms are non-negative, $\rho_1=\rho_k=0$. Therefore from (36) it follows that

$$\left(1+\frac{\lambda'}{1-\lambda'}\right)\lambda-\rho_k=\frac{\lambda'}{1-\lambda'}=\left(1+\frac{\lambda'}{1-\lambda'}\right)\lambda',$$

which contradicts (30), since $\rho_k \geqq 0$.[2] Consequently $V[\lambda'/(1-\lambda')] \notin R^*$.

On the basis of known properties of convex sets (see, for example, [7]), on the segment joining the interior point $V(0)$ of the set R^* with the point $V[\lambda'/(1-\lambda')]$ exterior to the set, there is one boundary point of the set. All of its coordinates obviously are positive so that this point can be taken for the sought for one.

We take the point which we have constructed (which we denote by V) and some (relative) interior point of R^* which we denote by U.

Let $V^* \in A^{-1}V$ and $U^* \in A^{-1}U$ be some pre-images of these points in the space E which belong to the simplex spanned by the unit basis vectors of this space, and $\mu^V_{|\mathfrak{R}|}$ and $\mu^U_{|\mathfrak{R}|}$ be the measures corresponding to these points on $\langle S_{|\mathfrak{R}|}, \mathfrak{S}_{|\mathfrak{R}|}\rangle$. In correspondence with (27) we form the families of measures $\mu^V_{\mathfrak{R}}$ and $\mu^U_{\mathfrak{R}}$. It is clear that these families are consistent.

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[2] We note that the contradiction in the system (30), (34)—(37) occurs only if a relation of type (36) is introduced. The system consisting of relations (30), (34), (35) ,and (37) is non-contradictory. Consequently, if n is such that the set $\mathbf{E}^*$ is empty, then this system can be solved. But the smallest number in $\mathbf{E}^*$ is 6. This means that if $2^n < 6$ (i.e. $n=2$) the proof which we have given does not work. In the final analysis this is why we are fortunate enough to be able to construct for any pair of random variables (independently of their domains of definition) their joint distribution, for example, the one for which our random variables are independent.

We now form the family of measures

$$\mu_{\mathfrak{K}} = (1+\varepsilon)\mu_{\mathfrak{K}}^{V} - \varepsilon\mu_{\mathfrak{K}}^{U}$$

(by the positivity of the coordinates of U and V, for sufficiently small $\varepsilon > 0$, $\mu_{\mathfrak{K}}$ will be a family of measures). By paragraph **1.22** the family $\mu_{\mathfrak{K}}$ will be consistent. But it cannot be extended, since, by well known properties of convex sets, $(1+\varepsilon)V - \varepsilon U \notin R^*$.

5°. The proof that $Z_n \notin \mathfrak{T}$ for $n \geqq 3$ is carried out without difficulty by constructing an example, which differs from that given in the introduction only in that $n-2$ copies must be taken of one of the random variables (say ξ_1).

Thus the class $\mathfrak{T}$ of complexes satisfies properties 1°—5°, of Theorem 3.2 and therefore coincides with the class $\mathfrak{R}$ of all regular complexes.

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CONSISTENT FAMILIES OF MEASURES AND THEIR EXTENSIONS

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(*Summary*)

Let Σ be a family of Borel fields of subsets of a set S and $\mu_{\mathfrak{S}}$ probabilistic measures on measurable spaces $\langle\mathfrak{S}, S\rangle$, where $\mathfrak{S} \in \Sigma$. The family of measures $\mu_{\mathfrak{S}}$, $\mathfrak{S} \in \Sigma$, is denoted by μ_{Σ}.

The measures $\mu_{\mathfrak{S}_1}$ and $\mu_{\mathfrak{S}_2}$ are said to be consistent if $\mu_{\mathfrak{S}_1}(A) = \mu_{\mathfrak{S}_2}(A)$ for any $A \in \mathfrak{S}_1 \cap \mathfrak{S}_2$. If any pair of measures of the family μ_{Σ} is consistent, the family itself is referred to as consistent.

The consistent family μ_{Σ} is said to be extendable if there is a measure $\mu_{[\Sigma]}$ on the measurable space $\langle[\Sigma], S\rangle$ consistent with each measure of μ_{Σ} ($[\Sigma]$ is the smallest Borel field containing all $\mathfrak{S} \in \Sigma$).

For the purposes of the theory of games the following special case of extendability is important. Let $\mathfrak{K}$ be a finite complete complex and M the set of its vertices. Let a finite set S_a correspond to each vertex a of $\mathfrak{K}$ and the set $S_A = \prod_{a\in A} S_a$ to each subset $A \subset M$. Let

$$\mathfrak{S}_K = \{X_K : X_K = Y_K \times S_{M-K},\quad Y_K \subset S_K\}, \qquad K \in \mathfrak{K};$$

μ_K is a measure on $\langle\mathfrak{S}_K, S_M\rangle$ and $\mu_{\mathfrak{K}}$ is the family of all such measures. The extendability of the family $\mu_{\mathfrak{K}}$ is closely related with the combinatorial properties of the complex $\mathfrak{K}$.

Any maximal face of the complex $\mathfrak{K}$ is said to be an extreme face if it has proper vertices (i.e. such vertices which do not belong to any other maximal face of $\mathfrak{K}$). If T is an extreme face of $\mathfrak{K}$ the complex $\mathfrak{K}^*$ obtained by removing from $\mathfrak{K}$ all proper vertices of T with their stars is said to be

a normal subcomplex of $\mathfrak{K}$. A complex $\mathfrak{K}$ is said to be regular if there is a sequence

$$\mathfrak{K} = \mathfrak{K}_0 \supset \mathfrak{K}_1 \supset \cdots \supset \mathfrak{K}_n$$

of subcomplexes of $\mathfrak{K}$ where $\mathfrak{K}_i$ is a normal subcomplex of $\mathfrak{K}_{i-1}$, $i = 1, \cdots, n$, and the last member vanishes.

The main results of the paper consists in the following statement.

Theorem. *The regularity of the complex $\mathfrak{K}$ is a necessary and sufficient condition of extendability of any consistent family of $\mu_{\mathfrak{K}}$ of measures.*

Cd I have used the topological terminology of the translation of Alexandrov's *Combinatorial Topology,* since the Russian edition of this text is the basic topological reference in this paper.