An Expanded Reflection-Coefficient Equation for Transmission-Line Junctions

When the familiar reflection coefficient is applied to a discontinuity in a transmission line, erroneous conclusions may result. W7WKB offers a revised formula and demonstrates how reflections become zero when a match is achieved.

By Roger Sparks, W7WKB

It is very unusual to see two knowledgeable men quarrel about technical matters in Amateur Radio publications, but that is exactly what has happened with a series of three articles from Steven Best, VE9SRB,[1] and one from Walt Maxwell, W2DU.[2] This unusual occurrence prompted me to study both articles and learn more about transmission lines and reflections than I really wanted to know! Nonetheless, it was very interesting, and I think my results are worth sharing.

Rather than siding with either of these gentlemen, I will present a different way of looking at the situation with an expanded equation to find the reflection coefficient for a discontinuity. The result for a ¼-wave matching section will be the same as Steven Best presented except for the canceling waves. The concept of complete reflection is more like Walt Maxwell's but without the idea of open or short circuits created by wave interference.

Begin with Basics

Our radio transmitters send a wave down a wire to the antenna. We will begin by forming an idea of how that invisible wave might look if we could see it.

My favorite analogy is to look at a canal or channel filled with water. The canal is very long and narrow, filled with some depth of water. Initially quiet, we see waves generated when we drop a rock into the water. The waves travel each way from the point of impact but leave the water surface quiet at the point of impact. If it were not for friction and if the canal were infinitely long, the waves could travel forever in both directions, carrying the energy from the initial splash to distant regions. The waves have a velocity and frequency, and carry energy just like electromagnetic waves traveling on a wire.

We can place our transmitter at the center of an infinitely long wire (I am not sure how we would find the center!) and generate waves which go in each direction. We would be creating a series of matching troughs and peaks paired as they fly off down the wire in opposite directions. By paired, I mean that one trough going to the

[1]Notes appear on page 19.

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left will be paired with a peak going to the right. The wave motion will be away from the initial point but the current for each trough-peak pair will be in the same direction. See Figure 1A.

Next, we can take our infinitely long wire and bend it into a U with the transmitter at the bend. We have now formed a transmission line. A wave generated at the bend in the U will travel down each side of the line with current going in opposite directions. See Figure 1B. In this situation the magnetic fields from the two currents cancel, as do the electrical fields, so long as you are measuring them from some distance away. If you measure really close to either wire, you will find the fields very much present.

If we use our transmission-line configuration but use a battery to generate the electrical pulse, we can ask "How much current would flow down the line when we make the battery connection?" If the line is infinitely long, will an infinitely large current flow? No, it turns out that a wave front is formed that charges the capacity between the wires to the battery voltage. The wave just travels along the wire charging the wire to the battery voltage at some rate of current. The speed of the wave is the speed of light (in air) or the velocity of propagation in a cable or insulated line. This velocity limit allows the capacity of the line to be charged only so fast, which means that a ratio of voltage to current can be found. Because the ratio of voltage to current is resistance, every transmission line will have a characteristic resistance.

The familiar equation for the resistance or impedance of a transmission line is

$$Z = \sqrt{LC}$$

where Z is the impedance, L the inductance, and C the capacity. The equation can also be written as

$$Z = \frac{1}{cC}$$

where c is the velocity of the wave, and C is the capacity per unit distance. This second form of the equation provides a rich insight into how the wave moves to fill the available capacity of the transmission line as it travels, and gives mathematical form to the previous wave description.

When an electrical wave travels down a wire in the real world, it always reaches a destination or end. How does the source (the battery or transmitter) know that the wave has reached the end of the wires? Feedback to the source cannot travel faster than the speed of light so the source just keeps putting power into the transmission line until the *reflected wave* arrives back at the source. That means that two waves are on the line at one time if we consider the reflected wave as a new wave as if it were from a second source.

When the wave is reflected from an open end, the current reverses but the voltage retains polarity. When measured, we see that the forward and reflected currents oppose but the voltages add. In terms of impedance, an open ended transmission line would measure capacitive impedance.[3]

If the wave travels to a short circuit, then an inverted wave is reflected, with the reflected current adding to the forward current but the voltages opposing and canceling. Thus you measure a high current but no voltage at end of a shorted transmission line. (Off subject, but this is why a loop antenna always has a low voltage point when measured equal distance from each side of the feed point.) A short-circuited transmission line would measure as an inductive impedance.[3]

Both reflected and source waves will have the impedance of the transmission line carrying them. If the wave travels to a resistance (terminated with a resistance), there will be no reflection *if the resistance is the same as the characteristic impedance of the line*. There will be a partial reflection of an open line type (capacitive) if the termination is a resistance higher than the line impedance, and a partial reflection of a short circuit type (inductive) if the termination resistance is less than the line impedance.

A transmission line terminated with a matching resistor appears to be infinite in length because there are no reflections. On the other hand, a mismatched line has reflections and is said to be discontinuous (or to contain a discontinuity).

Preparing to Derive a General Reflection Coefficient

Now let's ask a hard question: What will the reflected voltage be if some part of the power is delivered to a resistor?

As we work toward an answer, we will begin by selecting a circuit or visual model. The reflected wave travels back down the wire it just came

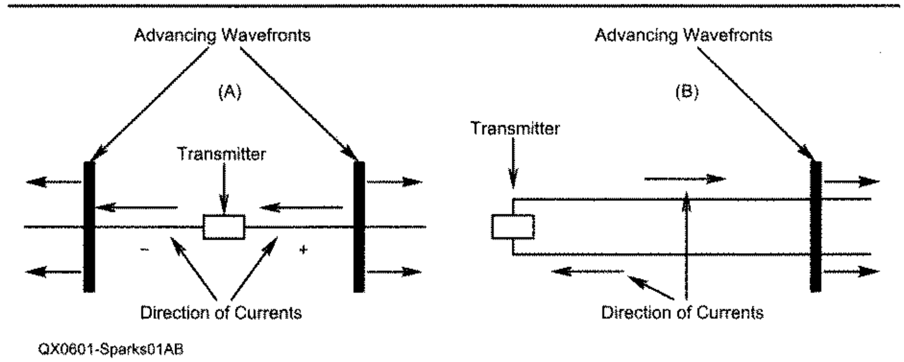


Figure 1 — (A) A transmitter at the center of an infinitely long wire generating wave pulses that go in each direction creates a series of matching troughs and peaks paired as they fly off down the wire in opposite directions. At B, the ends of the wire have been folded together to make it a feed line.

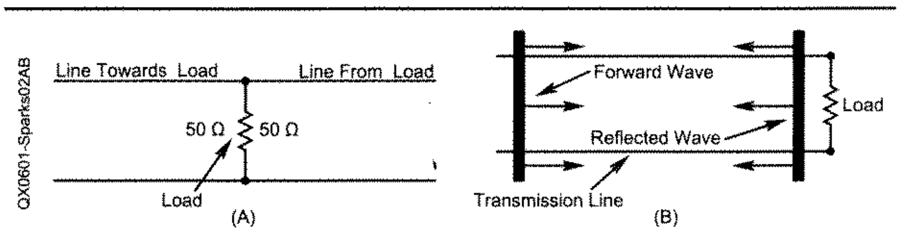


Figure 2 — Which circuit is the correct representation for a reflected wave? It depends upon how the waves interact. If there was no interaction at all, then circuit A would be correct. By measuring voltages and currents, we find that the two waves interact in a manner to result in equal voltage at all junctions, but the current will divide between two (or more) resistive paths with the result that current into the junction equals current leaving the junction. The division of leaving current depends upon both the resistance of each possible path and upon the incoming current from each possible path. Circuit B is the correct representation.

from. Will the circuit look like Figure 2A or 2B? If we use circuit 2A to build our mathematical model, the line picked to carry the reflection would look like a second resistance placed in parallel to the load resistance. In fact, circuit 2A is the circuit for a resistor placed into a transmission line. The line carrying the reflected wave has the same impedance as the source line, so the parallel combination of load and reflection carrying transmission line would always result in an inductive reflection.

Based on measurements, we can see that something else is happening when reflections occur. The reflected voltage or current adds or subtracts from the source power depending upon the resistive ratios. "Circuit" 2B must be used. It doesn't look like a circuit does it? It is more like a graphical description of the wave movement. This is how we will depict two waves flowing in opposite directions on a transmission line.

The familiar antenna reflection coefficient equation was derived from the "circuit" in Fig 2B.

$$\frac{Vfr}{Vf} = \frac{\frac{Zl}{Zf} - 1}{\frac{Zl}{Zf} + 1} \qquad \text{(Eq 1)}$$

where *Vfr* is the reflected part of the forward voltage, *Vf* is the forward voltage, *Zf* is the line impedance and *Zl* is the load impedance.

This equation was used very effectively by Steve Best in his article as he traced the multiple reflections that ultimately result in a stable transmission-line condition.[1]

The "circuit" in Figure 2B works fine for a transmission line terminated in either a resistor or antenna, but it must be expanded to include wave fronts coming from two directions if there is a discontinuity within the transmission line. Power coming from a second source will change the amount of primary power reflected from the discontinuity. Figure 3 is a representation of the wave flow from two directions at a discontinuity; it shows both forward and reflected waves.

We can use the familiar equation (Eq 1) for inline discontinuities (as Steven Best did), but the result leads us to the conclusion that a reflected wave from the discontinuity combines with a reflected wave from the antenna, which then cancels under matched conditions. We are left with the decidedly unsatisfactory notion that positive power cancels negative power. The power just seems to disappear, which we know to be impossible, from the laws of physics.

We can remove this unsatisfactory notion of canceled power by deriving the antenna reflection factor from a more general condition that includes power coming from a second direction on a conductor. To do this, we must break from the traditional premise that waves traveling in opposite directions pass through each other without effect. We substitute a premise that the waves always interact so that the resultant measurable voltage or current at any point is the sum of the voltages or currents of all the waves traveling in any direction on any one conductor.

If you are reluctant to give up the premise that opposite traveling waves pass without effect, look carefully at Figure 1A and Figure 1B to consider the differences between the two "circuits." If the reflected wave were to pass without effect, circuit (A) would apply. Instead, consider how waves travel when one wave rides "on top" of the other wave. Comfort may come after you understand how the expanded reflection coefficient is developed and actually works.

The general version of the reflection coefficient is developed in the sidebar "A Derivation of the Expanded Voltage-Ratio Equation."

The assumption that the voltage at any point is the sum of four waves rather than two deserves some explanation. When we consider a discontinuity, waves may source from both the right and left sides and meet at the point of consideration. A reflection will occur at the discontinuity for both waves. Thus, at the discontinuity, we have source waves arriving from both directions and reflected waves moving away in both directions, four waves. The voltage at the discontinuity is the sum of the two waves on each respective side, which becomes a single voltage at the single examination point.

Each reflected wave contains energy from both source waves but no instrument can identify from which source wave the energy actually came. Yes, it can be mathematically traced, but for all practical purposes, the

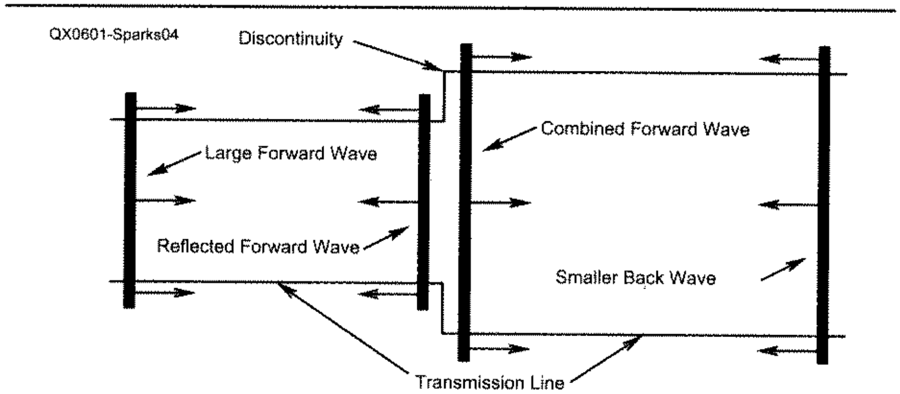


Figure 3 — A representation of the wave flow from two directions at a discontinuity; it shows both forward and reflected waves.

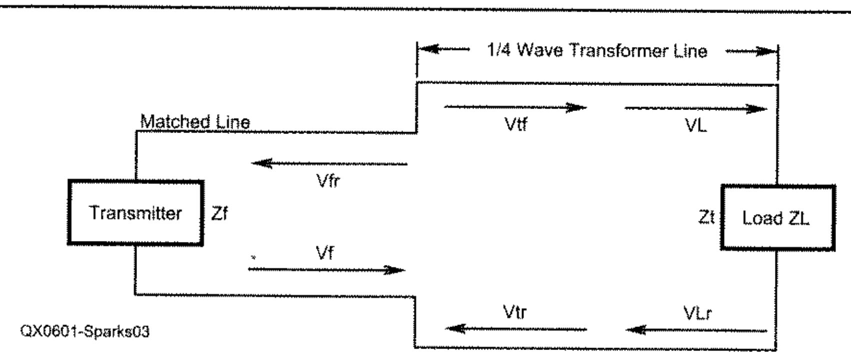


Figure 4 — Matched transmission line using 1/4 wave transformer.

source waves have been reformed into two new waves with a life of their own. The waves have been re-formed.

Here is the general reflection coefficient equation which was developed in the sidebar,

$$\frac{Vfr}{Vf} = \frac{\frac{Zl}{Zf} + 2\frac{Vlr}{Vf} - 1}{\frac{Zl}{Zf} + 1} \qquad \text{(Eq 2)}$$

where *Vlr* is the voltage reflected from the load and *Vf* is the forward source voltage.

One of the first things to notice about Eq 2 is that it reduces to Eq 1 if there is no reflected power. This is to be expected if it is a more comprehensive version of Eq 1.

The second thing to notice is that Eq 2 reduces to the pre-existing load reflection coefficient if there is no discontinuity in the transmission line. This is also to be expected if Eq 2 is a more comprehensive version of Eq 1.

Using the Expanded Reflection Coefficient

An analysis of the frequently used 1/4 wave matching transformer is a good way to show how the expanded reflection coefficient is used. I have attempted to use an example with the same characteristics as used in both the Best and Maxwell articles. This makes comparisons to the previous articles easier. Figure 4 shows the symbols used in the analysis. The results are shown in spreadsheet format as Table 1.

Assume that the impedance of the source line is 50 Ω, the impedance of the load is 150 Ω, and the transforming line is 86.6 Ω. The input power will be 100 W, which makes the forward voltage in the source line to be 70.711 V. The entire system will be assumed to be without losses so that we can focus on the principles of the problem.

The beginning source wave, *Vf*, will travel to the input of the 1/4-wave transformer where it has the first reflection. Part of the wave, *Vtf*, will continue forward to the load where it will divide between load, *Vl*, and backward reflection, *Vlr*. The second part of the source wave, *Vf*, will reflect from the input of the 1/4-wave section back to the source and will be identified as *Vfr*. (The effects of *Vfr* on the source will be ignored in this analysis for sake of simplicity.) The part of the wave reflected back from the load, *Vlr*, is relabeled *Vtr* so that *Vlr* = *Vtr* when we move to the next wave sequence. This is done to clarify the sequence of events as we trace the forward wave. The entire sequence of events is recorded on line 1 of Table 1.

When the leading edge of the reflected part of the initial wave reaches the input of the quarter wave line (when *Vtr* first contacts the leading edge of the second half wave *Vf* at the discontinuity), a new wave is formed at the input to the 1/4-wave line. Now we will use Eq 2 to find a new reflection coefficient that accounts for the power input from *Vtr*. The sequence of voltages for the second 1/2-wave is recorded on line 2 of Table 1.

This sequence of successive 1/2-wave events continues and is recorded on lower lines in Table 1 until the reflection coefficient *Vfr/Vf* becomes zero. At that point, stability is reached.

Steven Best will be gratified to notice that the steady state power excess in the 1/4-wave transformer is 7.736 W, the same as he discovered. Walt Maxwell will be glad to see that there is no power reflected back toward the source from the input of the 1/4-wave transformer once a steady state is reached. All of the power is properly directed by means of voltage division among resistive loads.

It is appropriate to emphasize the role of the 1/4-wave transformer in this example. In the 1/4-wave transformer, the leading wave edge has traveled exactly 1/2 wavelength before a portion of it returns to the input as a reflected wave (having entered the transformer, traveled 1/4 wavelength to the load, and 1/4 avelength back to the input point). On arrival back at the input, the leading edge "encounters" the second half of the wave, which is always inverted from the first half wave. We must account for this inversion by reversing the sign for *Vtr*. This reversal allows the reflection coefficient to reduce to zero after several half waves have occurred.

We have already mentioned that in the steady state, the transforming section of the 1/4-wavelength line always contains more power than the matched section of line (per unit

Table 1

Results of a spreadsheet to calculate the ratio of reflected voltage to source voltage (power is also calculated) using Eq 2 to analyze a 1/4-wavelength impedance transformer. A stable reflection factor, Vfr/Vf = 0, is reached after five reflection events. The voltage symbols for this spreadsheet/table are shown in Figure 4. The power associated with each voltage has a prefix "P"and a suffix identical to the associated voltage "V."

Input

Source impedance	50 Ω
Load impedance	150 Ω
Matching impedance	86.6 Ω
Source voltage	70.711

Calculations

Input Pulse Number	*Vtr*	*Vfr/Vf*	*Vfr*	*Pfr*	*Vtf*	*Ptf*	*Vlr/Vl*	*Vlr*	*Plr*	*Vl*	*P*	*Sum P*
1	0.000	0.268	18.946	7.179	89.657	92.822	0.268	24.025	6.665	113.682	86.157	100.001
2	24.025	0.019	1.358	0.037	96.094	106.629	0.268	25.750	7.656	121.844	98.973	106.666
3	25.750	0.001	0.096	0.000	96.556	107.657	0.268	25.873	7.730	122.430	99.927	107.657
4	25.873	0.000	0.005	0.000	96.589	107.731	0.268	25.882	7.736	122.472	99.996	107.731
5	25.882	–0.000	–0.002	0.000	96.592	107.736	0.268	25.883	7.736	122.475	100.001	107.736

A Derivation of the Expanded Voltage-Ratio Equation

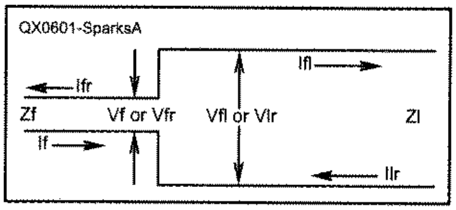
A junction in a real transmission line often carries waves traveling in both directions. What is the reflection factor — expressed as a ratio of reflected voltage to forward voltage — when power is flowing in both directions at a junction between two mismatched transmission lines?

Assumptions

1. Power entering the junction is equal to power leaving the junction.
2. Current entering the junction is equal to current leaving the junction.
3. The voltage developed at the junction is common to all conductors making the junction.
4. The voltage at the junction is the sum of the forward and reflected voltages for both the source and load sides of the junction. These voltages could be measured by a directional voltmeter.

Derivation

Refer to Figure A for a diagram and symbol description.



Zf = Impedance of the transmission line carrying forward power
Zl = Impedance of the transmission line carrying reflected power
Vf = Voltage of the forward wave
Vfr = Voltage of the reflected portion of the forward wave
Vlr = Voltage of the reflected wave traveling rearward from the load
Vfl = Voltage of the re-formed wave traveling forward to the load
If = Forward current
Ifr = Reflected current from junction
Ilr = Reflected current from the load
Ifl = Forward current of the re-formed wave traveling forward to the load

Based on our assumptions, we have:

$$Vf + Vfr = Vfl + Vlr \quad \text{(Eq S1)}$$

Voltage on the forward side equals voltage on load side, and

$$If + Ilr = Ifr + Ifl \quad \text{(Eq S2)}$$

current flowing into the junction equals current flowing out of it.

First, we find another definition for *Vfl* to reduce the number of unknown terms. The basic transmission line impedance is $Z = V / I$, where *V* is the input voltage and *I* is the input current. Therefore, we can say

$$Vfl = Ifl\,Zl$$

We can define *Vfl* in terms of *Zf* and *Zl* by using Eq S2 and the basic impedance relationship to substitute and rearrange terms.

$$Vfl = Zl(If + Ilr - Ifr)$$

$$Vfl = Zl\left(\frac{Vf}{Zf} + \frac{Vlr}{Zl} - \frac{Vfr}{Zf}\right) \quad \text{(Eq S3)}$$

The reflection ratio is defined as *Vfr / Vf*. We can substitute, using Eq S1, to get

$$\frac{Vfr}{Vf} = \frac{Vfl + Vlr - Vf}{Vf}$$

Next, multiply both sides by *Vf* and substitute *Vfl* by using Eq S3.

$$\frac{Vf\,Vfr}{Vf} = Vlr - Vf + Zl\left(\frac{Vf}{Zf} + \frac{Vlr}{Zl} - \frac{Vfr}{Zf}\right)$$

Carry out the multiplication

$$\frac{Vf\,Vfr}{Vf} = Vlr - Vf + \frac{Zl\,Vf}{Zf} + \frac{Zl\,Vlr}{Zl} - \frac{Zl\,Vfr}{Zf}$$

then simplify and combine terms.

$$Vfr = 2Vlr - Vf + \frac{Zl\,Vf}{Zf} - \frac{Zl\,Vfr}{Zf}$$

Divide both sides by *Vf*.

$$\frac{Vfr}{Vf} = \frac{2\,Vlr}{Vf} - 1 + \frac{Zl}{Zf} - \frac{Zl\,Vfr}{Zf\,Vf}$$

Group all the terms containing *Vfr* on the left side and factor out *Vfr / Vf*.

$$\frac{Vfr}{Vf}\left(\frac{Zl}{Zf} + 1\right) = \frac{Zl}{Zf} - 1 + \frac{2Vlr}{Vf}$$

Leave only *Vfr / Vf* on the left side to get the voltage reflection ratio:

$$\frac{Vfr}{Vf} = \frac{\frac{Zl}{Zf} - 1 + 2\left(\frac{Vlr}{Vf}\right)}{\frac{Zl}{Zf} + 1} \quad \text{(Eq S4)}$$

Equation S4 is the voltage reflection ratio for a transmission line discontinuity with power coming from two directions. Notice that the ratio is dependent upon the ratio of reflected load voltage (from the load) to the forward voltage, as well as the ratio of impedances on each side of the junction. Notice also that the ratio becomes the familiar

$$\frac{Vfr}{Vf} = \frac{\frac{Zl}{Zf} - 1}{\frac{Zl}{Zf} + 1} \quad \text{(Eq S5)}$$

if the reflected voltage from the load is zero.

length). The reflected wave is entirely reformed during each half cycle to contain just enough reflected power (extracted from the power going to the antenna) to cause the natural power division at the discontinuity to be zero in the source direction. At the same time, the reflected power is re-reflected from the discontinuity back to the antenna in phase with the source power. As a result, the transforming section always contains power from both the source wave and the reflected wave. In the example, the excess power is 7.736 W.

It is tempting to picture the reflections in the matching section as waves bouncing between mirrors. I would argue that it is more intellectually correct to say that the reflected wave is actually re-forming at each end using the power from both the forward and previously reflected waves. I must admit that the vision of mirrors has an appeal that is seductive, but it does not conjure an accurate conceptual framework.

Conclusion

We can take some of the mystery out of transmission-line reflections by using the expanded version of the reflection coefficient. The problem reduces to a sequence of voltage divisions that are easily calculated.

Acknowledgment

I would like to thank Mr Jon Pollock, KØZN, for reviewing the text and making many constructive suggestions.

Notes

[1]S. R. Best, VE9SRB, "Wave Mechanics of Transmission Lines," Part 1, *QEX*, Jan/Feb 2001, pp 3-8; Part 2, Jul/Aug 2001, pp 34-42; Part 3, Nov/Dec 2001, pp 43-50.

[2]W. Maxwell, W2DU, "A Tutorial Dispelling Certain Misconceptions Concerning Wave Interference in Impedance Matching," *QEX*, Jul/Aug 2004, pp 43-50.

[3]Whether the measured impedance of a transmission line is capacitive or inductive also depends on the distance between the measurement point and the line termination. For a purely resistive load, the first point of reversal is ¼-wavelength toward the source from the load. If the transmission line is long enough, additional reversals will occur each ¼ wavelength closer to the source from the first reversal point.

Roger was born in Jan 1937 and obtained his first Amateur Radio license in 1954. He graduated from Wallace High School in Wallace, Idaho, in 1955, and obtained an Agricultural Engineering degree from the University of Idaho in 1959. From 1959 through May 1962, Roger served in the US Navy, achieving the rank of Lieutenant JG. From September 1962 through February 1963, he was in General Dynamics' Centaur program. From March 1963 to present, he has worked in irrigated farming at Second Century Farms, Inc, at Ellensburg, Washington.

From 1974 to present, he has been an elected Commissioner of Kittitas County Public Utility District, providing electrical power.

In about 1989, Roger was elected to the Board of Directors of the Washington Public Power Supply System. In about 1998, he was elected to the Executive Board of Energy Northwest, which operates the Columbia Nuclear Generating Station at Richland, Washington.

Roger's hobbies include Amateur Radio, Amateur Seismology and the study of physics and mechanics. He has previously published "The C-T Quad" (in QST *and* The 1988 ARRL Antenna Anthology*) and "The Super Sloper" (Dec 1995* QST*).* □□