SYMMETRIC PRODUCTS

Preliminary Report

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Statement of the problem

In this article we study a problem which was discovered by students in a developmental mathematics course at Bronx Community College. They noticed that $(12)(21) = 252$, $(221)(122) = 26962$, and $(112)(211) = 23632$. That is, they noticed that if a number is multiplied by the number obtained by writing its digits backward, then the product is sometimes symmetric in the digits. Not every backward product is symmetric, however, since $(13)(31) = 403$. It turns out that there is a simple condition on the digits of a given number which determines whether its backward product is symmetric.

First let's recall some common terminology. The *tens digit representation* of a natural number $a = a_n a_{n-1} \cdots a_2 a_1 a_0$ represents the $n + 1$ digits of a. Here a_0 is the unit digit, a_1 is the tens digit, a_2 is the hundreds digits, and so on. Thus, we write a natural number a uniquely as

$$a = \sum_{i=0}^{n} a_i 10^i \quad \text{where} \quad 0 \leq a_i < 10 \quad \text{and} \quad a_n > 0.$$

The *length* of $a = a_n a_{n-1} \cdots a_2 a_1 a_0$ is the number of digits, $n + 1$.

Given the tens digit representation of a natural number $a = a_n a_{n-1} \cdots a_2 a_1 a_0$ let us say a is *symmetric* if $a_i = a_{n-i}$ for $0 \leq i < \frac{n}{2}$. For example, the numbers 252, 5225, 14341, and 20044002 are symmetric. Let us also call $(a_n a_{n-1} \cdots a_2 a_1 a_0)(a_0 a_1 a_2 \cdots a_{n-1} a_n)$ the *backward product* of a. For example, (123)(321) is the backward product of 123 and (9347)(7439) is the backward product of 9347. In this article we discuss the following conjecture.

Proposition. *Let $a = a_n a_{n-1} \cdots a_2 a_1 a_0$ be the tens digit representation of a natural number a. Then the backward product of a is symmetric if, and only if, $\sum a_i^2 < 10$.*

According to the proposition, the backward product (102002)(200201) of 102002 will be symmetric because the sum of squares of the digits of 102002 is less than 10. On the other hand, the backward product (10001022)(22010001) of 10001022 will not be symmetric because the sum of squares of the digits of 10001022 is 10.

In the next section we will show why the condition $\sum a_i^2 < 10$ on the digits of a is clearly sufficient for the backward product to be symmetric. Next, we will prove the necessity of the condition on the digits when the length of the backward product of a is odd. However, in general the length of the backward product can also be even. For example, $(17)(71) = 1207$ and $(119)(911) = 108409$. Computer experiments have shown that the length of the backward product of $a < 10^6$ is odd if the backward product is symmetric. The author is still uncertain why this is true for all natural numbers a, but his understanding of the problem is sketched in the last section.

Sufficiency of the digits condition

The sufficiency of the condition on the digits in the proposition is easy to see after reflecting for a moment on the tens digit representation of the backward product of a natural number $a = a_n a_{n-1} \cdots a_2 a_1 a_0$. Let us define

$$\delta_i = \sum_{j=0}^{i} a_j a_{n-i+j} \quad \text{and} \quad \delta_{2n-i} = \delta_i \quad \text{for} \quad 0 \leq i \leq n.$$

More concretely,

$$\begin{aligned}
\delta_0 &= a_0 a_n \\
\delta_1 &= a_0 a_{n-1} + a_1 a_n \\
\delta_2 &= a_0 a_{n-2} + a_1 a_{n-1} + a_2 a_n \\
\delta_3 &= a_0 a_{n-3} + a_1 a_{n-2} + a_2 a_{n-1} + a_3 a_n \\
&\vdots \\
\delta_n &= a_0^2 + a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2
\end{aligned}$$

and $\delta_{2n-i} = \delta_i$ for $0 \leq i \leq n$. Then evidently

$$\left(\sum_{i=0}^{n} a_i 10^i\right)\left(\sum_{i=0}^{n} a_{n-i} 10^i\right) = \sum_{i=0}^{2n} \delta_i 10^i.$$

In other words, the backward product of a is equal to $\sum \delta_i 10^i$. To compute the tens digit representation of a product one uses the division algorithm, inductively defining nonnegative integers c_i and d_i satisfying $0 \leq d_i < 10$ and

$$\begin{aligned}
\delta_0 &= 10c_0 + d_0 \\
\delta_1 + c_0 &= 10c_1 + d_1 \\
\delta_2 + c_1 &= 10c_2 + d_2 \\
&\vdots \\
\delta_2 + c_{2n-3} &= 10c_{2n-2} + d_{2n-2} \\
\delta_1 + c_{2n-2} &= 10c_{2n-1} + d_{2n-1} \\
\delta_0 + c_{2n-1} &= 10c_{2n} + d_{2n}.
\end{aligned}$$

Now, in general there are two possibilities: either $c_{2n} = 0$ or $c_{2n} > 0$. In the case of $c_{2n} = 0$ the tens digit representation of the backward product of a is $d_{2n} d_{2n-1} \cdots d_2 d_1 d_0$, whereas in the case of $c_{2n} > 0$ the tens digit representation of the backward product of a is $c_{2n} d_{2n} d_{2n-1} \cdots d_2 d_1 d_0$. Evidently, the length of the backward product of a is odd if, and only if, $c_{2n} = 0$.

With these observations the proof of the following lemma is straightforward.

Lemma 1. *Let $a = a_n a_{n-1} \cdots a_2 a_1 a_0$ be the tens digit representation of a natural number a. Then the backward product of a is symmetric if $\sum a_i^2 < 10$.*

Proof. The Schwarz inequality implies that for $0 \le i \le n$ we have

$$\delta_i = \sum_{j=0}^{i} a_j a_{n-i+j} \le \sqrt{\sum_{j=0}^{i} a_j^2}\sqrt{\sum_{j=0}^{i} a_{n-i+j}^2} \le \sum_{i=0}^{n} a_i^2.$$

Since $\delta_i = \delta_{2n-i}$ for $0 \le i \le n$ and $\delta_n = \sum a_i^2$ it follows that $\delta_i \le \delta_n$ for all i. Thus, $\delta_n = \sum a_i^2 < 10$ implies that $\delta_i < 10$ for all i. It follows from the division algorithm above that each $c_i = 0$ and hence $\delta_i = d_i$ for all $0 \le i \le 2n$. Therefore, the backward product is symmetric because $\delta_{2n-i} = \delta_i$. ■

Necessity of the digits condition in the odd case

According to the proof of Lemma 1, the length of the backward product of a is odd when the condition $\sum a_i^2 < 10$ on the digits is satisfied. Thus, the following lemma is a partial converse to Lemma 1.

Lemma 2. *Let $a = a_n a_{n-1} \cdots a_2 a_1 a_0$ be the tens digit representation of a natural number a. If the length of the backward product of a is odd and the backward product of a is symmetric, then $\sum a_i^2 < 10$.*

Proof. Since the length of the backward product of a is odd, its tens digit representation is $d_{2n} d_{2n-1} \cdots d_2 d_1 d_0$. If in addition the backward product is symmetric, then $d_{2n-i} = d_i$ for $0 \le i \le n$. We will prove that $c_i = c_{2n-1-i} = 0$ for $0 \le i \le n$ by induction on i.

First note that the division algorithm above gives

$$\begin{aligned} \delta_0 &= 10c_0 + d_0 \\ \delta_0 + c_{2n-1} &= d_0 \end{aligned}$$

so that subtracting gives $10c_0 + c_{2n-1} = 0$. Therefore the induction hypothesis holds for $i = 0$ because the integers c_i are nonnegative. Now assume the induction hypothesis holds for $0 < k < n$. Then the division algorithm above gives

$$\begin{aligned} \delta_{k+1} &= 10c_{k+1} + d_{k+1} \\ \delta_{2n-k-1} + c_{2n-k-2} &= d_{k+1} \end{aligned}$$

so that subtracting gives $10c_{k+1} + c_{2n-k-2} = 0$. Since the c_i are nonnegative it follows that $c_{k+1} = c_{2n-k-2} = 0$ which is the induction hypothesis for $k + 1$.

Therefore, $c_n = c_{n-1} = 0$ which implies that $\delta_n = d_n < 10$. This completes the proof because $\delta_n = \sum a_i^2$. ■

Some remarks on the even case

Lemma 1 and Lemma 2 state that if $a = a_n a_{n-1} \cdots a_2 a_1 a_0$ is the tens digit representation of a natural number a, and if the length of the backward product is odd, then the backward product of a is symmetric if, and only if, $\sum a_i^2 < 10$. To complete the proof of the proposition it is therefore sufficient to show that the length of the backward product is odd whenever the backward product is symmetric. Naturally, one would like to assume the backward product is symmetric and $c_{2n} > 0$, and then try to obtain a contradiction. To date the author has been unable to find a contradiction, although computer experiments for $a < 10^6$ strongly suggest that one should expect this.

If the length of the backward product of a is even and the backward product is symmetric, then the equations from the division algorithm become

$$\begin{aligned}
\delta_0 &= 10c_0 + d_0 \\
\delta_1 + c_0 &= 10c_1 + d_1 \\
\delta_2 + c_1 &= 10c_2 + d_2 \\
&\vdots \\
\delta_n + c_{n-1} &= 10c_n + d_n \\
\delta_{n-1} + c_n &= 10c_{n+1} + d_n \\
&\vdots \\
\delta_2 + c_{2n-3} &= 10c_{2n-2} + d_3 \\
\delta_1 + c_{2n-2} &= 10c_{2n-1} + d_2 \\
\delta_0 + c_{2n-1} &= 10d_0 + d_1.
\end{aligned}$$

All efforts to obtain a contradiction from these equations have failed. The problem seems to involve some delicate number theoretic questions related to which numbers can be represented by the quadratic forms δ_i.

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