THE UNIVERSAL COVER OF $\mathrm{SL}(2,\mathbb{R})$ AND ITS REPRESENTATIONS

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Contents

Introduction

Let $\mathrm{SL}(2,\mathbb{R})$ be the set of all 2×2 matrices over the real numbers with determinant 1. This expository article describes a representation theory for the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ which is motivated by a localization theory due to A. Beilinson and J. Bernstein [1]. We will explicitly describe the irreducible representations using objects familiar to the average calculus student—the complex plane, vector spaces, the algebra of 2×2 matrices, the derivative of a rational polynomial function—and some elementary facts about Lie group representations and algebraic $\mathcal{D}$-modules.

Historically, research on the representation theory of semisimple Lie groups has often required a careful analysis of SL_2 phenomena. The representation theory of $\mathrm{SL}(2,\mathbb{R})$, for example, arises naturally in the representation theory of any semisimple Lie group with finite center [5]. Likewise, the computations for the universal cover of $\mathrm{SL}(2,\mathbb{R})$ in this article have proven important in some recent developments on representations of semisimple Lie groups with infinite center [10]. Thus, this article may serve as an introduction to some aspects of current mathematical research.

The strategy pursued here will be familiar to disciples of the theory of Harish-Chandra sheaves [7], although some modifications are necessary for our purposes. While the objects we will consider in this article are quite concrete, the ideas and constructions can be explained more succinctly in full generality. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let σ be an involution of $\mathfrak{g}$. If $\mathfrak{h}$ is the abstract Cartan algebra of $\mathfrak{g}$, then for each $\lambda \in \mathfrak{h}^*$ one

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constructs a homogeneous sheaf of associative algebras $\mathcal{D}_\lambda$ which is locally isomorphic to the sheaf $\mathcal{D}_X$ of local differential operators on X. Now let $\mathfrak{k}$ be the set of vectors in $\mathfrak{g}$ fixed by σ. For representation theory it is useful to consider a connected algebraic group K with a morphism $\varphi : K \to \operatorname{Int}(\mathfrak{g})$ of algebraic groups whose differential $d\varphi$ maps the Lie algebra of K isomorphically onto $\mathfrak{k}$. Representations are commonly realized as global sections of certain $\mathcal{D}_\lambda$-modules which also have the structure of a quasi-coherent K-equivariant $\mathcal{O}_X$-module and satisfy a certain compatibility condition on these two actions. When the usual compatibility condition for Harish-Chandra sheaves is “weakened” as in this article, then one obtains so-called weakened Harish-Chandra sheaves.

It is possible to produce nontrivial weakened Harish-Chandra sheaves by a very natural geometric construction. Let Q be a K-orbit in the flag variety X, fix a point $x \in Q$, and let $\mathfrak{b}_x$ be the Borel subalgebra of $\mathfrak{g}$ corresponding to x. Choose a σ-stable Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ contained in $\mathfrak{b}_x$, and denote by $s : \mathfrak{h}^* \to \mathfrak{c}^*$ the specialization at x determined by the isomorphisms $\mathfrak{c} \to \mathfrak{b}_x/\mathfrak{n}_x \to \mathfrak{h}$, where $\mathfrak{n}_x$ is the nilpotent radical of $\mathfrak{b}_x$. If R is the set of roots for the pair $(\mathfrak{g}, \mathfrak{c})$ and R^+ is the set of positive roots defined by $\mathfrak{b}_x$, then the specialization s identifies R and R^+ with sets of abstract roots Σ respectively Σ^+ in $\mathfrak{h}^*$. An advantage of this perspective is that the triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ does not depend on the specialization s. Write ρ for the half-sum of positive roots in Σ.

Now let τ be the sheaf of local sections of an irreducible K-homogeneous vector bundle $E \to Q$; hence τ is a coherent K-equivariant $\mathcal{O}_Q$-module. If S is the stabilizer in K of x, then τ is uniquely determined by the action η of S on the geometric fiber $T_x(\tau)$. Fix $(\lambda, \omega) \in \mathfrak{h}^* \times \mathfrak{k}^*$ and assume ω vanishes on $[\mathfrak{k}, \mathfrak{k}]$. The sheaf τ is compatible with (λ, ω) if

$$s(\lambda + \rho)(\xi) = \eta(\xi) + \omega(\xi), \quad \xi \in \mathfrak{k} \cap \mathfrak{c}.$$

In this event we obtain a weakened Harish-Chandra sheaf as follows. If $i : Q \to X$ is the natural immersion, then τ has the structure of a $\mathcal{D}^i_\lambda$-connection. Thus, the direct image functor

$$i_+ : \mathcal{M}_{coh}(\mathcal{D}^i_\lambda) \longrightarrow \mathcal{M}_{coh}(\mathcal{D}_\lambda)$$

from the category of coherent $\mathcal{D}^i_\lambda$-modules to the category of coherent $\mathcal{D}_\lambda$-modules defines a coherent $\mathcal{D}_\lambda$-module

$$\mathcal{I}(Q, \tau) = i_+(\tau)$$

which is also endowed with the structure of a K-equivariant $\mathcal{O}_X$-module. By the compatibility condition, the differential of the K-action and the action of $\mathfrak{k}$ induced by the $\mathcal{D}_\lambda$-module action on $\mathcal{I}(Q, \tau)$ differ by the linear form $\omega \in \mathfrak{k}^*$.

All of this will be illustrated in subsequent sections for the complex semisimple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of 2×2 complex matrices with trace zero, the goal being to describe the irreducible representations of the universal cover G_0 of $\mathrm{SL}(2, \mathbb{R})$. For readers unacquainted with algebraic $\mathcal{D}$-modules and the localization theory of $\mathcal{U}_\theta$-modules, some relevant facts are listed in Appendix A.

Now suppose G_0 is an arbitrary connected semisimple Lie group with center Z_0. When Z_0 is finite, the irreducible representations of G_0 have been classified by Langlands using analytic methods [6] and by Vogan and Zuckerman using algebraic methods [11]. Very loosely speaking, when Z_0 is finite these methods involve constructing certain “standard” representations which have finite length from certain “tempered” representations and then obtaining all representations as composition factors. Classifying the tempered representations and understanding the composition series of the standard representations is therefore critical. Moreover, when Z_0 is finite, the geometric construction sketched above with $\omega = 0$ is known to produce (conventional) Harish-Chandra sheaves $\mathcal{I}(Q, \tau)$ whose global sections $\Gamma(X, \mathcal{I}(Q, \tau))$ are "dual" in some sense to the analytic construction of Langlands and algebraic construction of Vogan-Zuckerman [4]. Via this correspondence a number of known, but seemingly subtle results proved with analytic or algebraic methods can often be easily explained when interpreted in the language of Harish-Chandra sheaves. This leads one to hope that other open questions in representation theory can be answered using geometric methods.

Using analytic or algebraic methods, for example, some results have proven difficult to extend to the case in which the center Z_0 of G_0 is infinite. However, the global sections $\Gamma(X, \mathcal{I}(Q, \tau))$ of the *weakened* Harish-Chandra sheaves $\mathcal{I}(Q, \tau)$ constructed above for an arbitrary linear form $\omega \in \mathfrak{k}^*$ which vanishes on $[\mathfrak{k}, \mathfrak{k}]$ produce representations of G_0 when the center Z_0 is possibly infinite. Thus, for the representation theory of G_0 with infinite center one might try to use the geometry of the flag variety of its complexified Lie algebra $\mathfrak{g}$ to 1) determine precisely when a standard module $\mathcal{I}(Q, \tau)$ is irreducible, 2) determine which standard modules are tempered (having global sections which are tempered modulo the center), and 3) compute the composition factors of $\mathcal{I}(Q, \tau)$ and their multiplicities. The first two general goals were achieved in [10]; the last goal is the subject of the author's ongoing research. In this article, however, we will give a solution to all three of these problems when G_0 is the universal cover of SL$(2, \mathbb{R})$.

1. The flag variety of $\mathfrak{sl}(2, \mathbb{C})$

In this section we study in detail the structure of the flag variety X of $\mathfrak{sl}(2, \mathbb{C})$. Although X is just the projective line $\mathbb{P}^1$ in this simple example, the geometry of X considered as a homogeneous space is quite rich. This underlying complexity together with the old familiarity with which even freshman calculus students regard these objects, and the importance this example plays in the general representation theory of semisimple Lie groups, accounts for the allure of this field for many researchers.

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ be the set of all complex 2×2 matrices with trace zero. For matrices X and Y in $\mathfrak{g}$ put $[X, Y] = XY - YX$ to give $\mathfrak{g}$ the structure of a complex semisimple Lie algebra. If we define $\sigma : \mathfrak{g} \to \mathfrak{g}$ by

$$\sigma(X) = JXJ^{-1} \quad \text{where} \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then evidently $\sigma([X, Y]) = [\sigma(X), \sigma(Y)]$ for all $X, Y \in \mathfrak{g}$ and $\sigma^2 = 1$; hence σ is an involution of $\mathfrak{g}$. Let $\{E, F, H\}$ denote the standard basis of $\mathfrak{g}$ where

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$. A simple computation shows that $\mathfrak{k} = \mathbb{C}H$ is the set of vectors in $\mathfrak{g}$ fixed by σ. Finally, consider the following subgroups of

$G = \mathrm{PSL}(2,\mathbb{C})$:

$$T = \left\{ \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \mid x \in \mathbb{C}^* \right\},$$
$$N = \left\{ \begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix} \mid w \in \mathbb{C} \right\},$$
$$B = \left\{ \begin{bmatrix} x & y \\ 0 & x^{-1} \end{bmatrix} \mid x \in \mathbb{C}^*, y \in \mathbb{C} \right\},$$
$$\overline{N} = \left\{ \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \mid z \in \mathbb{C} \right\}.$$

Then $\mathfrak{t} = \mathfrak{k}$, $\mathfrak{n} = \mathbb{C}E$, $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$, and $\overline{\mathfrak{n}} = \mathbb{C}F$ are the Lie algebras of T, N, B, and $\overline{N}$ respectively.

The *flag variety* of $\mathfrak{g}$ is the set X of all Borel subalgebras of $\mathfrak{g}$. Since every Borel subalgebra of $\mathfrak{g}$ is conjugate in G to $\mathfrak{b}$, and the normalizer of $\mathfrak{b}$ in G is B, it follows that X can be identified with the smooth projective variety G/B. Furthermore, if $\mathbb{P}^1$ denotes the complex projective line with homogeneous coordinates $[z_0, z_1]$, then in this example we can identify X with $\mathbb{P}^1$ by letting G act on $\mathbb{P}^1$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} az_0 + bz_1 \\ cz_0 + dz_1 \end{bmatrix}.$$

That is, if $0 = [1,0]$, then B is the stabilizer in G of 0 and $G/B \to \mathbb{P}^1, gB \mapsto g.0$ is an isomorphism. For a point $x \in \mathbb{P}^1$ we often denote the corresponding Borel subalgebra in X by $\mathfrak{b}_x$; for example, $\mathfrak{b}_0 = \mathfrak{b}$ corresponds to $0 \in \mathbb{P}^1$ and $\mathfrak{b}_\infty = \mathfrak{t} \oplus \overline{\mathfrak{n}}$ corresponds to $\infty = [0,1]$.

As indicated in the Introduction, we want to consider a connected algebraic group K and a morphism $\varphi : K \to \mathrm{PSL}(2,\mathbb{C})$ of algebraic groups whose differential $d\varphi$ maps the Lie algebra of K isomorphically onto $\mathfrak{k}$. Let us take $K = T$. Evidently $\mathfrak{g}$ contains exactly two K-conjugacy classes of σ-stable Cartan subalgebras; they are represented by

$$\mathfrak{t} = \left\{ \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix} \mid h \in \mathbb{C} \right\} \quad \text{and} \quad \mathfrak{c} = \left\{ \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix} \mid h \in \mathbb{C} \right\}.$$

Note that $\sigma = 1$ on $\mathfrak{t}$ so that $\mathfrak{t}$ is the *fundamental* Cartan, and $\sigma = -1$ on $\mathfrak{c}$ so that $\mathfrak{c}$ is a *split* Cartan.

Under our identification of the flag variety X with $\mathbb{P}^1$, the K-orbits in X are $\mathbb{C}^* = K.1 = \mathbb{P}^1 - \{0, \infty\}$, $\{0\}$, and $\{\infty\}$. We know that $\mathfrak{t} \subset \mathfrak{b}_0$, and for this reason we say *the K-orbit $\{0\}$ is attached to* $\mathfrak{t}$. Similarly, since $\mathfrak{t} \subset \mathfrak{b}_\infty$ we say *the K-orbit $\{\infty\}$ is attached to* $\mathfrak{t}$.

We also say *the K-orbit $\mathbb{C}^*$ is attached to* $\mathfrak{c}$. To see why let us put

$$X_\alpha = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad Y_\alpha = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad H_\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so that $\{X_\alpha, Y_\alpha, H_\alpha\}$ is an SL_2-basis of $\mathfrak{g}$; that is, $[H_\alpha, X_\alpha] = 2X_\alpha$, $[H_\alpha, Y_\alpha] = -2Y_\alpha$, and $[X_\alpha, Y_\alpha] = H_\alpha$. If R denotes the roots in $\mathfrak{c}^*$ for the pair $(\mathfrak{g}, \mathfrak{c})$, then the specialization at $1 = [1,1] \in \mathbb{C}^*$ maps $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ into $(\mathfrak{c}^*, R, R^+)$. Under our identification of the flag variety with $\mathbb{P}^1$, we see that the Borel subalgebra $\mathfrak{b}_1 = \mathfrak{c} \oplus \mathfrak{g}_\alpha$ corresponds to 1, where $\mathfrak{g}_\alpha = \mathbb{C}X_\alpha$; hence $\mathfrak{c} \subset \mathfrak{b}_1$.

2. Algebraic $\mathcal{D}$-modules

Let $\mathcal{D}$ be a homogeneous twisted sheaf of differential operators on a smooth algebraic variety. Some fundamental constructions from the general theory of algebraic $\mathcal{D}$-modules are sketched in Appendix A. In this section the details are worked out explicitly for the flag variety of $\mathfrak{sl}(2,\mathbb{C})$. The details of this example are well known. However, they are included here for completeness since a new twist will be introduced in Section 3 when we construct some weakened Harish-Chandra sheaves.

Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ and let X be the flag variety of $\mathfrak{g}$ as described in Section 1. If f is a function on X, then G acts by $(g.f)(x) = f(g^{-1}.x)$ for $g \in G$ and $x \in X$. Differentiating this action, $\mathfrak{g}$ acts by

$$(\xi.f)(x) = \frac{d}{dt} f(\exp(-t\xi).x)\bigg|_{t=0} \quad \text{for} \quad \xi \in \mathfrak{g} \text{ and } x \in X.$$

For $\lambda \in \mathfrak{h}^*$ we can describe trivializations of the G-homogeneous sheaf of differential operators $\mathcal{D}_\lambda$ on X on a canonical open cover of $\mathbb{P}^1$. Let $\{U_0, U_\infty\}$ be the open cover of $\mathbb{P}^1$ consisting of the sets $U_0 = \mathbb{P}^1 - \{\infty\}$ and $U_\infty = \mathbb{P}^1 - \{0\}$ with respective coordinates $\mathbb{C} \to U_0$, $z \mapsto [1,z]$ and $\mathbb{C} \to U_\infty$, $w \mapsto [w,1]$. Note that $U_0 = \overline{N}.0$ and $U_\infty = N.\infty$, and under our identification of the flag variety with $\mathbb{P}^1$ we see that if we specialize at 0, then H corresponds to the dual root $\alpha^\vee$, whereas if we specialize at ∞, then H corresponds to $-\alpha^\vee$. Given $\lambda \in \mathfrak{h}^*$ let us abbreviate $t = \alpha^\vee(\lambda)$, and furthermore write $\pi : \mathfrak{g} \to \Gamma(X, \mathcal{D}_\lambda)$ for the morphism induced by λ.

Let us construct trivializations of $\mathcal{D}_\lambda$ on U_0. Since $U_0 = \overline{N}.0$ then evidently $\pi(F) = -\partial$ where $\partial = \partial/\partial z$ in the coordinate $\mathbb{C} \to U_0$, $z \mapsto [1,z]$. Now write $\pi(H) = a\partial + b$ where $a, b \in \mathbb{C}[z]$. Then a short calculation using the relation $[H,F] = -2F$ implies that $a = 2z + a_0$ and $b = b_0$ where a_0 and b_0 are constants. Such a calculation is performed as follows: first verify that

$$[a\partial + b, \partial] = -(\partial a)\partial - \partial b;$$

then the relation $[H,F] = -2F$ implies that

$$(\partial a)\partial + \partial b = 2\partial;$$

hence $\partial a = 2$ and $\partial b = 0$. (Subsequent calculations of this type will be left as an exercise.) On the other hand, in the geometric fiber of $\mathcal{D}_\lambda$ at 0, $\pi(H) - (t+1)$ acts by zero. So $a_0 = 0$ and $b_0 = t+1$. Thus $\pi(H) = 2z\partial + (t+1)$. Finally, write $\pi(E) = c\partial + d$ with $a, b \in \mathbb{C}[z]$ and perform a similar calculation using the relation $[H,E] = 2E$ to get $c = c_0 z^2$ and $d = d_0 z$ where c_0 and d_0 are constants; then use the relation $[E,F] = H$ to get $c_0 = 1$ and $d_0 = t+1$. To summarize,

$$\pi(E) = z^2\partial + (t+1)z, \quad \pi(F) = -\partial, \quad \pi(H) = 2z\partial + (t+1). \tag{1}$$

Next, let us construct trivializations of $\mathcal{D}_\lambda$ on U_∞. Recall that $U_\infty = N.\infty$, so evidently $\pi(E) = -\partial$ where $\partial = \partial/\partial w$ for the coordinate $\mathbb{C} \to U_\infty$, $w \mapsto [w,1]$. Using the relation $[H,E] = 2E$ gives $\pi(H) = (-2w + a_0)\partial + b_0$ where a_0 and b_0 are constants. Since H corresponds to $-\alpha^\vee$ in the specialization at ∞, then $\pi(H) + (t+1)$ acts by zero in the geometric fiber of $\mathcal{D}_\lambda$ at ∞; hence $a_0 = 0$ and $b_0 = -(t+1)$. Finally, if $\pi(F) = e\partial + f$

where $e, f \in \mathbb{C}[w]$ then the relations $[H, F] = -2F$ and $[E, F] = H$ imply that $e = w^2$ and $f = (t+1)w$. To summarize,

$$\pi(E) = -\partial, \quad \pi(F) = w^2\partial + (t+1)w, \quad \pi(H) = -2w\partial - (t+1). \tag{2}$$

The two trivializations are related on $\mathbb{C}^* = U_0 \cap U_\infty$ as follows. Since $w = \frac{1}{z}$ on $\mathbb{C}^*$ it follows that

$$\frac{\partial}{\partial w} = -z^2 \frac{\partial}{\partial z}.$$

Hence changing coordinates in (2) gives

$$\pi(E) = z^2\partial, \quad \pi(F) = -\partial + \frac{t+1}{z}, \quad \pi(H) = 2z\partial - (t+1) \tag{3}$$

on $\mathbb{C}^*$.

3. Standard modules

The constructions which follow produce nontrivial weakened Harish-Chandra sheaves for the triple $(\mathfrak{sl}(2,\mathbb{C}), K, \omega)$ where $\varphi : K \to \mathrm{PSL}(2,\mathbb{C})$ is the identity morphism onto the diagonal subgroup T and $\omega \in \mathfrak{k}^*$. Fix $(\lambda, \omega) \in \mathfrak{h}^* \times \mathfrak{k}^*$ and let Q be a K-orbit in the flag variety $X = \mathbb{P}^1$. Appendix A indicates the general procedure for constructing a standard module $\mathcal{I}(Q, \tau)$ when τ is the sheaf of local sections of an irreducible K-equivariant $\mathcal{O}_Q$-module compatible with (λ, ω). For this example, the standard modules are easy to describe explicitly.

Proposition 3.1. *The standard weakened Harish-Chandra sheaves $\mathcal{I}(Q, \tau)$ for the triple $(\mathfrak{sl}(2,\mathbb{C}), K, \omega)$ are of the form*

(i) $\mathcal{I}(\{0\}, t, \epsilon)$ *when* $t \in 2\mathbb{Z} + 1 + \epsilon$,
(ii) $\mathcal{I}(\{\infty\}, t, \epsilon)$ *when* $t \in 2\mathbb{Z} + 1 - \epsilon$,
(iii) $\mathcal{I}(\mathbb{C}^*, t, \epsilon)$,

where $t, \epsilon \in \mathbb{C}$ *and* $\mathrm{Re}\,\epsilon \in [0, 2)$.

Proof. For our computations we shall identify K with $\mathbb{C}^*$ so that the natural K-action on $\mathbb{P}^1$ is $\zeta.[z_0, z_1] = [\zeta z_0, z_1]$ for $\zeta \in K = \mathbb{C}^*$. With this action, the differential of φ maps $\zeta\partial_\zeta$ to $-\frac{1}{2}H$. Recall that for $\lambda \in \mathfrak{h}^*$ we put $\alpha^\vee(\lambda) = t$. For $\omega \in \mathfrak{k}^*$ also let $\omega(H) = \epsilon$.

First we consider the closed K-orbits. The stabilizer in K of 0 is $S = K$, and the irreducible representations of S are $\{\eta_n\}$ where $\eta_n(\zeta) = \zeta^n$, $n \in \mathbb{Z}$. Let τ_n be the irreducible K equivariant $\mathcal{O}_{\{0\}}$-module such that S acts by η_n on the geometric fiber $T_0(\tau_n)$. Then τ_n is compatible with (λ, ω) if, and only if,

$$t + 1 = -2n + \epsilon.$$

In this event, if $i : \{0\} \to \mathbb{P}^1$ is the natural immersion, then we write $\mathcal{I}(\{0\}, t, \epsilon) = i_+(\tau_n)$ for the standard module with standard data $(\{0\}, \tau_n)$. Thus, $\mathcal{I}(\{0\}, t, \epsilon)$ exists if, and only if,

$$t \in 2\mathbb{Z} + 1 + \epsilon.$$

A similar computation shows that a standard module $\mathcal{I}(\{\infty\}, t, \epsilon)$ exists if, and only if,

$$t \in 2\mathbb{Z} + 1 - \epsilon.$$

Next we consider the open K-orbit in X. The stabilizer of 1 in K is trivial and here $\mathfrak{k} \cap \mathfrak{b}_1 = \{0\}$. Therefore, for each (λ, ω) there is one, and only one irreducible K-equivariant $\mathcal{O}_{\mathbb{C}^*}$-module τ. More concretely, we see that $\tau = \mathcal{O}_{\mathbb{C}^*}$ as an $\mathcal{O}_{\mathbb{C}^*}$-module. If $i : \mathbb{C}^* \to \mathbb{P}^1$ is the natural immersion, then we write $\mathcal{I}(\mathbb{C}^*, t, \epsilon) = i_+(\tau)$ for the standard module with standard data $(\mathbb{C}^*, \tau)$. They exist for every $t, \epsilon \in \mathbb{C}$.

There are obvious isomorphisms between the various standard modules attached to (t, ϵ) and (t, ϵ') when $\epsilon - \epsilon' \in 2\mathbb{Z}$. Therefore, we can assume that the real part of ϵ is contained in $[0, 2)$. □

4. An irreducibility theorem

For the closed orbits, the standard modules $\mathcal{I}(\{0\}, t, \epsilon)$ and $\mathcal{I}(\{\infty\}, t, \epsilon)$ are all irreducible. On the other hand, the modules $\mathcal{I}(\mathbb{C}^*, t, \epsilon)$ may be reducible. Determining precise conditions for reducibility and describing the composition series when they are reducible is the goal of this article.

Although a formulation of our main result does not require additional notation, the following important observation turns out to be useful for the general representation theory of semisimple Lie groups. Define X_α and Y_α as in Section 1 and then put

$$Z_\alpha^+ = X_\alpha + Y_\alpha \quad \text{and} \quad Z_\alpha^- = X_\alpha - Y_\alpha.$$

Note that 1) the vector space $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is σ-invariant, 2) Z_α^+ and Z_α^- span the $+1$ and -1 σ-eigenspaces of $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ respectively, and 3) they satisfy the relations

$$\begin{aligned}[] [H_\alpha, Z_\alpha^+] &= 2Z_\alpha^-, \\ [H_\alpha, Z_\alpha^-] &= 2Z_\alpha^+, \\ [Z_\alpha^+, Z_\alpha^-] &= -2H_\alpha. \end{aligned}$$

It is easy to check that the only eigenbases of $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ which satisfy these relations are $\{Z_\alpha^+, Z_\alpha^-\}$ and $\{-Z_\alpha^+, -Z_\alpha^-\}$. Since $\sigma(X_\alpha) = Y_\alpha$ it follows from the construction of Z_α^+ that $\sigma(Z_\alpha^+) = Z_\alpha^+$; hence $Z_\alpha^+ \in \mathfrak{k}$. In fact, if

$$n_\alpha = \exp_K(\pi i Z_\alpha^+),$$

then n_α lies in the stabilizer S of 1 in K. For $\omega \in \mathfrak{k}^*$ let us also define

$$\alpha^+(\omega) = -\omega(Z_\alpha^+).$$

Now, let τ be an irreducible K-equivariant $\mathcal{O}_{\mathbb{C}^*}$-module compatible with $(\lambda, \omega) \in \mathfrak{h}^* \times \mathfrak{k}^*$ and denote by η the action of the stabilizer S on the geometric fiber $T_1(\tau)$. We say that τ *satisfies the* SL_2*-parity condition* if the spectrum of $\eta(n_\alpha)$ does not contain the numbers

$$-e^{\pi i \{\alpha^+(\omega) \pm \alpha^\vee(\lambda)\}}.$$

Theorem 4.1. *Let τ be the irreducible K-equivariant $\mathcal{O}_{\mathbb{C}^*}$-module compatible with (λ, ω). Then the standard module $\mathcal{I}(\mathbb{C}^*, \tau)$ is irreducible if, and only if, τ satisfies the* SL_2*-parity condition.*

Proof. Let s be a section of $\mathcal{I}(\mathbb{C}^*, \tau)$ on $\mathbb{C}^*$ on which K acts trivially; hence, if ν is the differential of the K-action, then $\nu(H)s = 0$. Recall that we defined $t = \alpha^\vee(\lambda)$ and $\alpha^+(\omega) = \epsilon$ because $Z_\alpha^+ = H$. By ω-compatibility the actions of π and ν differ by ω. Using (3) gives

$$2z\partial s - (t+1)s = \epsilon s,$$

and solving this equation we get $s = z^{\frac{1}{2}(t+1+\epsilon)}$. It follows that $\mathcal{I}(\mathbb{C}^*, \tau)|_{\mathbb{C}^*} = s\mathcal{O}_{\mathbb{C}^*}$ because τ is irreducible. Now put $s_p = z^{p+\frac{1}{2}(t+1+\epsilon)}$ for $p \in \mathbb{Z}$. According to (3) we have

$$\pi(E)s_p = \left[p + \frac{1}{2}(t+1+\epsilon)\right] s_{p+1},$$
$$\pi(F)s_p = \left[-p + \frac{1}{2}(t+1-\epsilon)\right] s_{p-1}.$$

Evidently $\mathcal{I}(\mathbb{C}^*, \tau)$ has a nontrivial quotient supported at 0 if, and only if, $\pi(F)s_p = 0$ for some $p \in \mathbb{Z}$, i.e.,

$$t \in 2\mathbb{Z} + 1 + \epsilon. \tag{4}$$

On the other hand, $\mathcal{I}(\mathbb{C}^*, \tau)$ has a nontrivial quotient supported at ∞ if, and only if, $\pi(E)s_p = 0$ for some $p \in \mathbb{Z}$, i.e.,

$$t \in 2\mathbb{Z} + 1 - \epsilon. \tag{5}$$

The proof is completed by comparing (4) and (5) with the definition of the SL_2-parity condition. □

It is important to note that the SL_2-parity condition as we have formulated it does not depend on the choice of $\pm Z_\alpha^+$. Also, this “natural” definition is useful in more general settings because it makes sense for any algebraic group K whose connected component is an n-fold cover of T. It turns out that some questions in the representation theory of an arbitrary connected semisimple Lie group G_0 (possibly with infinite center) reduce to this case. Although Theorem 4.1 is a particularly simple special case, its proof needs only minor modifications to generalize it to the case in which the connected component of K is an n-fold cover of T.

Recall that a basic general property of the standard module $\mathcal{I}(Q, \tau)$ is that it has a unique irreducible submodule $\mathcal{L}(Q, \tau)$. The proofs of Theorem 4.1 and Proposition 3.1 have the following corollary.

Corollary 4.2. *Consider a pair* $(t, \epsilon) \in \mathbb{C}^2$ *and assume* $\operatorname{Re} \epsilon \in [0, 2)$.

(i) *If* $t \in 2\mathbb{Z} + 1 + \epsilon$ *and* $\epsilon \notin \{0, 1\}$, *then there is a short exact sequence of weakened Harish-Chandra sheaves*

$$0 \to \mathcal{L}(\mathbb{C}^*, t, \epsilon) \to \mathcal{I}(\mathbb{C}^*, t, \epsilon) \to \mathcal{L}(\{0\}, t, \epsilon) \to 0.$$

(ii) *If* $t \in 2\mathbb{Z} + 1 - \epsilon$ *and* $\epsilon \notin \{0, 1\}$, *then there is a short exact sequence of weakened Harish-Chandra sheaves*

$$0 \to \mathcal{L}(\mathbb{C}^*, t, \epsilon) \to \mathcal{I}(\mathbb{C}^*, t, \epsilon) \to \mathcal{L}(\{\infty\}, t, \epsilon) \to 0.$$

(iii) *If* $t \in 2\mathbb{Z} + 1 \pm \epsilon$ *and* $\epsilon \in \{0, 1\}$*, then* $\mathcal{O}(t+1)$ *is the unique irreducible submodule of* $\mathcal{I}(\mathbb{C}^*, t, \epsilon)$ *and there is a short exact sequence of weakened Harish-Chandra sheaves*

$$0 \to \mathcal{O}(t+1) \to \mathcal{I}(\mathbb{C}^*, t, \epsilon) \to \mathcal{L}(\{0\}, t, \epsilon) \oplus \mathcal{L}(\{\infty\}, t, \epsilon) \to 0.$$

5. Classification of irreducible representations

Let G_0 be the universal covering group of SL$(2,\mathbb{R})$. In this section we will give a simple classification of the irreducible representations of G_0 using the geometry of the flag variety of $\mathfrak{sl}(2,\mathbb{C})$. The irreducible representations of G_0 were studied quite some time ago [9]. In this article, however, the irreducible representations of G_0 are realized as global sections of irreducible weakened Harish-Chandra sheaves.

To simplify our computations, consider the subgroup of SL$(2,\mathbb{C})$ given by

$$\mathrm{SU}(1,1) = \left\{ \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

It is well known that SU$(1,1)$ and SL$(2,\mathbb{R})$ are conjugate in SL$(2,\mathbb{C})$ and are therefore isomorphic. Hence, without loss of generality we can replace G_0 by the universal cover of SU$(1,1)$ in this discussion. The following simple facts account for our preference of SU$(1,1)$ over SL$(2,\mathbb{R})$. Let $\mathfrak{g}_0 = \mathfrak{su}(1,1)$ be the Lie algebra of G_0. Then its complexification is $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, and the composition of conjugations with respect to the real forms $\mathfrak{g}_0$ and $\mathfrak{su}(2)$ is just the involution σ defined in Section 1. Let $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$ so that $\mathfrak{k}$ is its complexification. Topologically, SU$(1,1) = S^1 \times \mathrm{U}$ where S^1 is the unit circle and $\mathrm{U} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the interior of the unit disk in $\mathbb{C}$. Thus, the fundamental group of G_0 is $\mathbb{Z}$ and it follows that the center Z_0 of G_0 is infinite cyclic. If K_0 is the connected Lie subgroup of G_0 with Lie algebra $\mathfrak{k}_0$, then $K_0 = \mathbb{R}$ and one checks that Z_0 is generated by

$$u = \exp_{K_0}(2\pi i H) \tag{6}$$

for $H \in \mathfrak{k}$ defined in Section 1.

By a *representation* of G_0 we mean a pair (π, V) where V is a Hilbert space, $\pi : G_0 \to \mathrm{Aut}(V)$ is a group homomorphism into the invertible linear transformations of V, and for each $v \in V$ the function $G_0 \to V$, $g \mapsto \pi(g)v$ is continuous. A subspace W of V is said to be *invariant* if $\pi(g)W \subset W$ for all $g \in G_0$, and we say V is *irreducible* if $V \neq \{0\}$ and the only closed invariant subspaces of V are $\{0\}$ and V itself. A homomorphism $\zeta : Z_0 \to \mathbb{C}^*$ of the center Z_0 is called a *central character*, and we say a representation (π, V) is ζ*-central* if Z_0 acts on V be the central character ζ.

By Schur's Lemma, every irreducible representation of G_0 is ζ-central for some central character ζ. Thus, we want to describe the irreducible ζ-central representations (π, V) of G_0 for each central character ζ. Recall that u in (6) generates Z_0; hence a central character ζ is defined by $\zeta(u) = e^{\pi i \epsilon}$ for some $\epsilon \in \mathbb{C}$. If $u_t = \exp_{K_0}(2\pi t i H)$ for $t \in \mathbb{R}$, then we can extend ζ to a character ω of K_0 by $\omega(u_t) = e^{\pi t i \epsilon}$ and check that $\omega(H) = \epsilon$. In this way $\nu = \pi \otimes \omega^{-1}$ defines an action of K_0 on V which descends to an action of K_0/Z_0. On the other hand, as a subgroup of PSL$(2,\mathbb{C})$ we have

$$K_0/Z_0 = \left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \mid \theta \in \mathbb{R} \right\},$$

and the complexification of K_0/Z_0 in $\mathrm{PSL}(2,\mathbb{C})$ is the subgroup K in Section 1 of diagonal complex matrices.

In general, if $\mathfrak{g}$ is a complex semisimple Lie algebra, we say a linear form $\lambda \in \mathfrak{h}^*$ is *strongly antidominant* if $\operatorname{Re}\alpha^\vee(\lambda) \leq 0$ for all $\alpha \in \Sigma^+$. For our example of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ it is clear that λ is strongly antidominant if, and only if, $\operatorname{Re} t \leq 0$. If X is the flag variety of $\mathfrak{g}$, then some general results of Harish-Chandra and some results related to the localization of $\mathcal{U}_\theta$-modules sketched in Appendix A show that an irreducible ζ-central representation (π, V) of G_0 can be attached to the global sections $\Gamma(X, \mathcal{L}(Q,t,\epsilon))$ for some $(t,\epsilon) \in \mathbb{C}^2$ where $\operatorname{Re} t \leq 0$ and $\operatorname{Re}\epsilon \in [0,2)$. Here Q is a K-orbit in X and $\mathcal{L}(Q,t,\epsilon)$ is the unique irreducible submodule of the standard module $\mathcal{I}(Q,t,\epsilon)$ in Proposition 3.1. However, the irreducible weakened Harish-Chandra sheaves $\mathcal{L}(Q,t,\epsilon)$ may have no global sections. The next result, which is a straightforward application of a vastly more general nonvanishing result in [10], essentially completes our classification.

Proposition 5.1. *Suppose* $\operatorname{Re} t \leq 0$.

(i) *If* $t \in 2\mathbb{Z}+1+\epsilon$, *then* $\Gamma(X, \mathcal{L}(\{0\},t,\epsilon)) \neq 0$.
(ii) *If* $t \in 2\mathbb{Z}+1-\epsilon$, *then* $\Gamma(X, \mathcal{L}(\{\infty\},t,\epsilon)) \neq 0$.
(iii) $\Gamma(X, \mathcal{L}(\mathbb{C}^*,t,\epsilon)) = 0$ *if, and only if,* $t=0$ *and* $\epsilon = 1$.

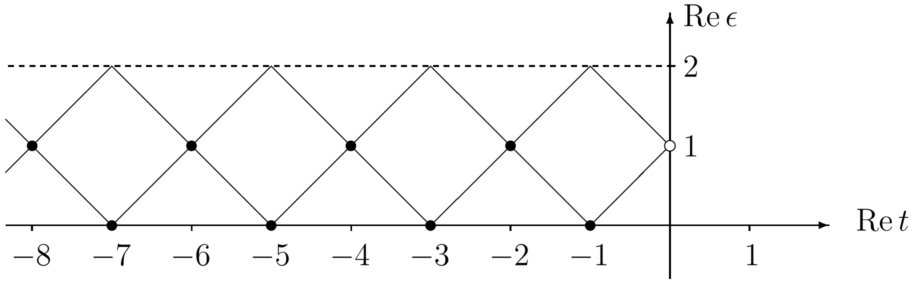

Figure 1. Nonvanishing of $\Gamma(X, \mathcal{L}(Q,t,\epsilon))$

Proposition 5.1 is illustrated in Figure 5. In this figure, the modules $\mathcal{L}(\{0\},t,\epsilon)$ exist on the lines with slope $+1$ and the modules $\mathcal{L}(\{\infty\},t,\epsilon)$ exist on the lines with slope -1; they both exist at each of the dots (open and closed). On the other hand, the standard modules $\mathcal{I}(\mathbb{C}^*,t,\epsilon)$ exist for all (t,ϵ) and by Theorem 4.1 they are reducible on the lines with slopes ± 1. The open dot at $(0,1)$ indicates that the global sections of $\mathcal{L}(\mathbb{C}^*,0,1)$ all vanish; all of the other modules $\mathcal{L}(\mathbb{C}^*,t,\epsilon)$ with $\operatorname{Re} t \leq 0$ have nontrivial global sections.

Let us relabel the irreducible representations of G_0 listed in Proposition 5.1. In light of Corollary 4.2, they are essentially of four types.

(I) For $n \in \mathbb{N}$ put $F_n = \Gamma(X, \mathcal{O}(-n))$. If $n > 0$, then F_n is the finite dimensional representation of dimension n.
(II) Put $D_{t,+\epsilon} = \Gamma(X, \mathcal{L}(\{0\},t,\epsilon))$ for $t \in 2\mathbb{Z}+1+\epsilon$, and $D_{t,-\epsilon} = \Gamma(X, \mathcal{L}(\{\infty\},t,\epsilon))$ for $t \in 2\mathbb{Z}+1-\epsilon$.
(III) Let $I_{t,\epsilon} = \Gamma(X, \mathcal{I}(\mathbb{C}^*,t,\epsilon))$.
(IV) Let $L_{t,\epsilon} = \Gamma(X, \mathcal{L}(\mathbb{C}^*,t,\epsilon))$.

Then Proposition 5.1 can be restated as follows.

Corollary 5.2. *If* G_0 *is the universal cover of* $\mathrm{SL}(2,\mathbb{R})$*, then every irreducible representation is equivalent to one of the following:*

(i) F_n *for* $n \in \mathbb{N}$,
(ii) $D_{t,\pm\epsilon}$ *for* $\operatorname{Re} t \leq 0$ *and* $t \in 2\mathbb{Z}+1\pm\epsilon$,
(iii) $I_{t,\epsilon}$ *for* $\operatorname{Re} t \leq 0$ *and* $t \notin 2\mathbb{Z}+1\pm\epsilon$,
(iv) $L_{t,\pm\epsilon}$ *for* $\operatorname{Re} t < 0$, $t \in 2\mathbb{Z}+1\pm\epsilon$, *and* $\epsilon \notin \{0,1\}$.

Furthermore, these representations are mutually inequivalent.

Next we describe the irreducible tempered representations of G_0. Let (π, V) be a representation of G_0 and let $\langle\cdot\,,\cdot\rangle$ be the inner product on V. Then for each $v, w \in V$ the continuous function

$$c_{v,w} : G_0 \longrightarrow \mathbb{C}, \quad g \mapsto \langle \pi(g)v, w\rangle$$

is called a *matrix coefficient* of V. If V is irreducible, then evidently each matrix coefficient descends to G_0/Z_0. The notion of a *tempered* representation is a certain growth condition on the matrix coefficients of V, and understanding the irreducible tempered representations is critical in the Langlands classification [6]. A representation (π, V) is *square integrable* (modulo the center) if it has a nonzero matrix coefficient which is square integrable on G_0/Z_0. Every square integrable representation is tempered, meaning that its matrix coefficients are rapidly decreasing in some sense. Although the asymptotic behavior of matrix coefficients is an "analytic" invariant, there is a simple, completely geometric interpretation of temperedness for weakened Harish-Chandra sheaves [10] which is based on its connection with the $\mathfrak{n}$-homology of V established in [3]. Applying these results to our example leads to the following fact.

Theorem 5.3. *Let* G_0 *be the universal cover of* $\mathrm{SL}(2,\mathbb{R})$.

(i) *The irreducible square integrable representations are* $D_{t,\pm\epsilon}$ *for* $t \in 2\mathbb{Z}+1\pm\epsilon$ *and* $\operatorname{Re} t < 0$.
(ii) *The other irreducible tempered representations are* $D_{0,\pm 1}$ *and* $I_{t,\epsilon}$ *for* $\operatorname{Re} t = 0$ *and* $\epsilon \neq 1$.

Thus, in Figure 5 the modules on the lines with slopes ±1 supported on $\{0\}$ and $\{\infty\}$ respectively have square integrable global sections when $\operatorname{Re} t < 0$; the other tempered representations supported on the closed orbits are the global sections of $\mathcal{L}(\{0\}, 0, 1)$ and $\mathcal{L}(\{\infty\}, 0, 1)$. The remaining tempered representations are the global sections of $\mathcal{L}(\mathbb{C}^*, t, \epsilon)$ for $\operatorname{Re} t = 0$ and $\epsilon \neq 1$; in Figure 5 this is the ϵ-axis minus the point $(0,1)$. It is interesting to note that for these points $\mathcal{I}(\mathbb{C}^*, t, \epsilon)$ is irreducible by Theorem 4.1. Thus, the irreducible tempered representations are global sections of a standard weakened Harish-Chandra sheaf. Finally, note that at the point $(0,1)$ the standard module $\mathcal{I}(\mathbb{C}^*, 0, 1)$ is reducible, and by Proposition 5.1 (iii) its unique irreducible submodule has no global sections. Therefore, Corollary 4.2 (iii) implies that $\Gamma(X, \mathcal{I}(\mathbb{C}^*, 0, 1))$ is the direct sum $D_{0,1} \oplus D_{0,-1}$ of the tempered representations attached to the closed orbits at the point $(0,1)$ in Figure 5.

This completes our study of the irreducible representations of the universal covering group of $\mathrm{SL}(2,\mathbb{R})$. It is interesting to recall, in passing, the well-known classification of the irreducible representations of $\mathrm{SL}(2,\mathbb{R})$ to see which "new" representations arise for its universal covering group. The center of $\mathrm{SL}(2,\mathbb{R})$ is $\{\pm I\}$ where I is the identity matrix. Every representation of $\mathrm{SL}(2,\mathbb{R})$ defines a representation of its universal cover G_0, and evidently an irreducible ζ-central representation of G_0 descends to a representation of $\mathrm{SL}(2,\mathbb{R})$ if, and only if, $\epsilon \in \{0,1\}$. Thus, referring to Figure 5 we see that the irreducible representations of $\mathrm{SL}(2,\mathbb{R})$ are essentially of three types.

(I) The finite dimensional representations F_n for $n > 0$.
(II) The "discrete series" representations $D_{n,\pm\epsilon}$ for $\epsilon \in \{0,1\}$, $n \in 2\mathbb{Z}+1+\epsilon$, and $n < 0$, and the "limits of discrete series" $D_{0,\pm 1}$.
(III) The "irreducible principal series" representations $I_{t,\epsilon}$ for $\epsilon \in \{0,1\}$, $t \notin 2\mathbb{Z}+1+\epsilon$, and $\operatorname{Re} t \leq 0$.

Appendix A. General $\mathcal{D}$-module constructions

Here we collect some basic facts on twisted sheaves of differential operators on a smooth algebraic variety [2], [7]. We shall briefly sketch an application of this theory to the localization theory of $\mathcal{U}_\theta$-modules [1], describe the general construction of standard weakened Harish-Chandra sheaves $\mathcal{I}(Q,\tau)$, and list some of their basic properties [8].

Let X be a smooth complex algebraic variety, $\mathcal{O}_X$ the structure sheaf of X, and $\mathcal{D}_X$ the sheaf of local differential operators on X. By a *twisted sheaf of differential operators* we mean a pair $(\mathcal{D}, i)$ where

(T1) $\mathcal{D}$ is a sheaf of associative algebras on X;
(T2) $i : \mathcal{O}_X \to \mathcal{D}$ is a morphism of algebras;
(T3) $\mathcal{D}$ is locally isomorphic to $\mathcal{D}_X$.

If $f : Y \to X$ is a morphism of smooth algebraic varieties and $\mathcal{D}$ is a twisted sheaf of differential operators on X, we put

$$\mathcal{D}_{Y\to X} = f^*(\mathcal{D}) = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}.$$

Denote by $\mathcal{D}^f$ the sheaf of those left $\mathcal{O}_Y$-module endomorphisms of $\mathcal{D}_{Y\to X}$ which commute with the right $f^{-1}\mathcal{D}$-action. Then $\mathcal{D}^f$ is a twisted sheaf of differential operators on Y; for example, $\mathcal{D}_X^f = \mathcal{D}_Y$.

When an algebraic group G acts transitively on X there is a simple classification of the G-homogeneous twisted sheaves of differential operators on X. Denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. Fix $x \in X$ and let B be the stabilizer of x in G. If τ denotes the natural action of $\mathfrak{g}$ on $\mathcal{O}_X$ then $\mathcal{U}^0 = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$ becomes a sheaf of associative algebras with multiplication defined by

$$(f \otimes \xi)(g \otimes \eta) = f\tau(\xi)g \otimes \eta + fg \otimes \xi\eta$$

where $f, g \in \mathcal{O}_X$, $\xi \in \mathfrak{g}$, and $\eta \in \mathcal{U}(\mathfrak{g})$. Let $\mathfrak{g}^0 = \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$, considered as an $\mathcal{O}_X$-submodule of $\mathcal{U}^0$. The natural commutator in $\mathcal{U}^0$ induces the structure of a sheaf of Lie algebras on $\mathfrak{g}^0$. The map τ extends to a homomorphism of $\mathfrak{g}^0$ into the sheaf of local vector fields $\mathcal{F}_X$ on X, and we denote by $\mathfrak{b}^0$ the kernel of τ. Then $\mathfrak{b}^0$ is a sheaf of ideals in $\mathfrak{g}^0$, and its geometric fiber $T_x(\mathfrak{b}^0) = \mathfrak{b}$ is just the Lie algebra of B. Thus, for each B-invariant linear form $\mu \in \mathfrak{b}^*$ we can construct a distinct G-equivariant $\mathcal{O}_X$-module morphism $\sigma_\mu : \mathfrak{b}^0 \to \mathcal{O}_X$. If $\varphi_\mu : \mathfrak{b}^0 \to \mathcal{U}^0$ is given by $\varphi_\mu(s) = s - \sigma_\mu(s)$, then im φ_μ generates a sheaf of two-sided ideals $\mathcal{I}_\mu$ in $\mathcal{U}^0$. Define

$$\mathcal{D}_{X,\mu} = \mathcal{U}^0/\mathcal{I}_\mu.$$

Then $\mathcal{D}_{X,\mu}$ is a *G-homogeneous twisted sheaf of differential operators* on X, and all homogeneous twisted sheaves of differential operators arise in this way.

There is a simple necessary and sufficient condition for an $\mathcal{O}_X$-module $\mathcal{V}$ to have the structure of a $\mathcal{D}_{X,\mu}$-module. We say $T \in \mathcal{E}nd(\mathcal{V}, \mathcal{V})$ is a *local differential endomorphism of order* $\leq n$ if

$$[\dots[[T, f_0], f_1], \dots, f_n] = 0$$

for any $n+1$ regular functions $f_0, f_1, \dots, f_n \in \mathcal{O}_X$. Denote by $\mathcal{D}iff(\mathcal{V}, \mathcal{V})$ the sheaf of all local differential endomorphisms of $\mathcal{V}$ (of any order). For example, note that $\mathcal{D}_X = \mathcal{D}iff(\mathcal{O}_X, \mathcal{O}_X)$. Given a linear map

$$\pi : \mathfrak{g} \longrightarrow \mathcal{D}iff(\mathcal{V}, \mathcal{V})$$

satisfying

$$\pi([\xi, \eta]) = [\pi(\xi), \pi(\eta)] \quad \text{for all} \quad \xi, \eta \in \mathfrak{g},$$

then one naturally obtains a morphism

$$\mathcal{U}^0 \longrightarrow \mathcal{D}iff(\mathcal{V}, \mathcal{V})$$

between sheaves of associative algebras. Fix $x \in X$ and let B denote the stabilizer in G of x. Let $\mu \in \mathfrak{b}^*$ be a B-invariant linear form, where $\mathfrak{b}$ is the Lie algebra of B. It follows easily that π defines the structure of a $\mathcal{D}_{X,\mu}$-module on $\mathcal{V}$ if, and only if,

$$\pi(\xi) = \mu(\xi).I \quad \text{for all} \quad \xi \in \mathfrak{b}.$$

Conversely, a $\mathcal{D}_{X,\mu}$-module structure on $\mathcal{V}$ determines such a π.

If $\mathcal{D}$ is a twisted sheaf of differential operators on X, then the opposite sheaf of algebras $\mathcal{D}^{opp}$ is again a twisted sheaf of differential operators on X. We can therefore view left $\mathcal{D}$-modules as right $\mathcal{D}^{opp}$-modules and vice versa. Formally, the category $\mathcal{M}^L_{qc}(\mathcal{D})$ of quasi-coherent left $\mathcal{D}$-modules is isomorphic to the category $\mathcal{M}^R_{qc}(\mathcal{D}^{opp})$ of quasi-coherent right $\mathcal{D}^{opp}$-modules. Hence we can freely use right or left modules depending on the particular situation.

Let $\mathcal{D}$ be a twisted sheaf of differential operators on X. Denote by $D^b(\mathcal{M}^R_{qc}(\mathcal{D}))$ or simply $D^b(\mathcal{D})$ the derived category of bounded complexes of quasi-coherent right $\mathcal{D}$-modules. The abelian category $\mathcal{M}^R_{qc}(\mathcal{D})$ embeds in $D^b(\mathcal{D})$ via the functor D which attaches to a right $\mathcal{D}$-module $\mathcal{W}$ the complex $D(\mathcal{W})$ which has $\mathcal{W}$ in degree zero and the zero module elsewhere. Also, for each $p \in \mathbb{Z}$ we have cohomology functors $H^p : D^b(\mathcal{D}) \to \mathcal{M}^R_{qc}(\mathcal{D})$.

Consider a morphism $f : Y \to X$ of smooth algebraic varieties. Given $\mathcal{W}^\bullet$ in $D^b(\mathcal{D}^f)$ we put

$$Rf_+(\mathcal{W}^\bullet) = Rf_*(\mathcal{W}^\bullet \overset{L}{\otimes}_{\mathcal{D}^f} \mathcal{D}_{Y\to X}).$$

so that one obtains the *direct image functor*

$$Rf_+ : D^b(\mathcal{D}^f) \longrightarrow D^b(\mathcal{D}).$$

When f is an immersion, its direct image f_+ functor has some useful properties.

Theorem A.1. *Suppose $f : Y \to X$ is an immersion of smooth algebraic varieties. Then*

(i) $H^p \circ Rf_+ = 0$ *for* $p < 0$*;*

(ii) Rf_+ *is the right derived functor of the functor*

$$H^0 \circ Rf_+ \circ D : \mathcal{M}^R_{qc}(\mathcal{D}^f) \longrightarrow \mathcal{M}^R_{qc}(\mathcal{D});$$

(iii) *if* $\mathcal{W} \in \mathcal{M}^R_{qc}(\mathcal{D}^f)$ *and* $V = \operatorname{supp} \mathcal{W}$*, then* $\operatorname{supp} R^0 f_+(\mathcal{W}) = \overline{V}$*;*

(iv) *if* f *is affine, then* $R^p f_+ = 0$ *for* $p > 0$*.*

In the case of an immersion $i : Y \to X$ we often abbreviate $R^0 i_+ = i_+$.

Theorem A.2. *If* Y *is a closed smooth subvariety of a smooth algebraic variety* X*,* $i : Y \to X$ *is the natural immersion, and* $\mathcal{D}$ *is a twisted sheaf of differential operators on* X*, then the functor*

$$i_+ : \mathcal{M}_{qc}(\mathcal{D}^i) \longrightarrow \mathcal{M}_{qc}(\mathcal{D})$$

establishes an equivalence of categories between $\mathcal{M}_{qc}(\mathcal{D}^i)$ *and the full subcategory of* $\mathcal{M}_{qc}(\mathcal{D})$ *consisting of modules supported in* Y*.*

For the remainder of this appendix we assume X is the flag variety of a complex semisimple Lie algebra $\mathfrak{g}$. That is, X is the set of all Borel subalgebras of $\mathfrak{g}$. For $x \in X$ denote by $\mathfrak{b}_x$ the corresponding Borel subalgebra, and let $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ be its nilpotent radical. The adjoint action of the group $G = \operatorname{Int}(\mathfrak{g})$ on $\mathfrak{g}$ endows X with the structure of a G-homogeneous space. The stabilizer in G of x is the Borel subgroup B of G with Lie algebra $\mathfrak{b}_x$; hence X can be identified with the smooth projective variety G/B.

If $\mathcal{B}$ denotes the G-homogeneous vector bundle of Borel subalgebras and $\mathcal{N}$ denotes its subbundle of nilpotent radicals, then $\mathcal{H} = \mathcal{B}/\mathcal{N}$ is a trivial G-homogeneous vector bundle on X. The set of global sections $\mathfrak{h} = \Gamma(X, \mathcal{H})$ has the structure of an abelian Lie algebra; we call $\mathfrak{h}$ the *abstract Cartan* of $\mathfrak{g}$. Now fix $x \in X$ and choose a Cartan subalgebra $\mathfrak{c} \subset \mathfrak{b}_x$. The isomorphisms $\mathfrak{c} \to \mathfrak{b}_x/\mathfrak{n}_x \to \mathfrak{h}$ define a *specialization* $s : \mathfrak{h}^* \to \mathfrak{c}^*$ which identifies the roots R in $\mathfrak{c}^*$ for the pair $(\mathfrak{g}, \mathfrak{c})$ with a set of roots Σ in $\mathfrak{h}^*$. Let Σ^+ denote the roots in Σ corresponding to the positive roots R^+ determined by $\mathfrak{b}_x$. Then the triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ does not depend on x or the choice of the Cartan $\mathfrak{c}$ in $\mathfrak{b}_x$. Let ρ be the half-sum of positive roots in Σ, and let W denote the Weyl group of Σ.

For each $\lambda \in \mathfrak{h}^*$ we define

$$\mathcal{D}_\lambda = \mathcal{D}_{X,\lambda+\rho},$$

where $\mathcal{D}_{X,\lambda+\rho}$ is the G-homogeneous twisted sheaf of differential operators on X described above. Then $\mathcal{D}_\lambda$ is characterized by the property that $\xi - (\lambda + \rho)(\xi)$ acts by zero in the geometric fiber of $\mathcal{D}_\lambda$ at x for all $\xi \in \mathfrak{h}$. By the work of Harish-Chandra, the Weyl group orbits θ in $\mathfrak{h}^*$ lie in one-to-one correspondence to maximal ideals J_θ in the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$. If $\lambda \in \theta$, then under this correspondence, the kernel of the natural map $\mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_\lambda)$ is simply $J_\theta \mathcal{U}(\mathfrak{g})$. Hence, $\Gamma(X, \mathcal{D}_\lambda) = \mathcal{U}_\theta$ where $\mathcal{U}_\theta = \mathcal{U}(\mathfrak{g})/J_\theta \mathcal{U}(\mathfrak{g})$.

Fix $\lambda \in \mathfrak{h}^*$ and let $\theta = W.\lambda$. Let $\mathcal{M}(\mathcal{U}_\theta)$ denote the category of $\mathcal{U}_\theta$-modules, and let $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ denote the category of quasi-coherent $\mathcal{D}_\lambda$-modules. The global sections functor

$$\Gamma : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{U}_\theta)$$

is left exact. Beilinson and Bernstein [1] defined the *localization functor*

$$\Delta_\lambda : \mathcal{M}(\mathcal{U}_\theta) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_\lambda)$$

by

$$\Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$$

for V in $\mathcal{M}(\mathcal{U}_\theta)$. Then Δ_λ is right exact and is a left adjoint functor to Γ. We say that λ is *antidominant* if $\alpha^\vee(\lambda)$ is not a strictly positive integer for any $\alpha \in \Sigma^+$. The following result is proved in [7].

Theorem A.3. *Let* $\lambda \in \mathfrak{h}^*$ *be antidominant and* $\theta = W.\lambda$.

- (i) *If* $\mathcal{V}$ *is an irreducible* $\mathcal{D}_\lambda$*-module, then either* $\Gamma(X, \mathcal{V})$ *is an irreducible* $\mathcal{U}_\theta$*-module or it is zero.*
- (ii) *If* V *is an irreducible* $\mathcal{U}_\theta$*-module, then there exists a unique irreducible* $\mathcal{D}_\lambda$*-module* $\mathcal{V}$ *such that* $\Gamma(X, \mathcal{V}) = V$.

We say that λ is *regular* if $\alpha^\vee(\lambda) \neq 0$ for all $\alpha \in \Sigma$.

Theorem A.4. *If* $\lambda \in \mathfrak{h}^*$ *is antidominant and regular and* $\theta = W.\lambda$, *then the localization functor*

$$\Delta_\lambda : \mathcal{M}(\mathcal{U}_\theta) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_\lambda)$$

is an equivalence of categories. Its inverse is Γ.

Thus, the study of irreducible $\mathcal{U}_\theta$-modules reduces to the problem of classifying all irreducible $\mathcal{D}_\lambda$-modules and the problem of describing those irreducible $\mathcal{D}_\lambda$-modules with no global sections.

Finally, we recall the construction of standard weakened Harish-Chandra sheaves and their basic properties. Let σ be an involution of $\mathfrak{g}$, and let $\mathfrak{k}$ be the set of vectors in $\mathfrak{g}$ fixed by σ. Consider a connected algebraic group K with a morphism $\varphi : K \longrightarrow \mathrm{Int}(\mathfrak{g})$ of algebraic groups whose differential $d\varphi$ maps the Lie algebra of K isomorphically onto $\mathfrak{k}$. If $\omega \in \mathfrak{k}^*$ is a linear form which vanishes on $[\mathfrak{k}, \mathfrak{k}]$, then we say that $\mathcal{V}$ is an *ω-weakened Harish-Chandra sheaf*, or simply a *weakened Harish-Chandra sheaf* if

- (W1) $\mathcal{V}$ is a coherent $\mathcal{D}_\lambda$-module;
- (W2) $\mathcal{V}$ is a K-equivariant $\mathcal{O}_X$-module;
- (W3) $\pi(\xi) = \nu(\xi) + \omega(\xi)$ for all $\xi \in \mathfrak{k}$.

Here π is induced by the $\mathcal{D}_\lambda$-action and ν is the differential of the K-action. Denote by $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \omega)$ the category of weakened Harish-Chandra sheaves. As with conventional Harish-Chandra sheaves (when $\omega = 0$), there is a very elegant classification of irreducible weakened Harish-Chandra sheaves.

Let Q be a K-orbit in X with immersion $i : Q \to X$. Fix $x \in Q$ and let $\mathfrak{b}_x$ be the corresponding Borel subalgebra of $\mathfrak{g}$. If S is the stabilizer in K of x, then the Lie algebra of S is $\mathfrak{k} \cap \mathfrak{b}_x$. Choose a Cartan subalgebra $\mathfrak{c}$ contained in $\mathfrak{b}_x$ and let $s : \mathfrak{h}^* \to \mathfrak{c}^*$ be a specialization at x. Given $\lambda \in \mathfrak{h}^*$, let μ denote the restriction of $s(\lambda + \rho)$ to $\mathfrak{k} \cap \mathfrak{c}$ so that $\mathcal{D}^i_\lambda = \mathcal{D}_{Q,\mu}$. Fix $(\lambda, \omega) \in \mathfrak{h}^* \times \mathfrak{k}^*$ and assume ω vanishes on $[\mathfrak{k}, \mathfrak{k}]$. If τ is an irreducible K-equivariant $\mathcal{O}_Q$-module, then we say that τ *is compatible with* (λ, ω) if

$$\mu(\xi) = \eta(\xi) + \omega(\xi), \quad \xi \in \mathfrak{k} \cap \mathfrak{c},$$

where η is the differential of the S-action on the geometric fiber $T_x(\tau)$. In this event, we call

$$\mathcal{I}(Q, \tau) = i_+(\tau)$$

the *standard module* with *standard data* (Q, τ); it is a nontrivial weakened Harish-Chandra sheaf. Proofs of the following results are analogous to the proofs of the corresponding results for conventional Harish-Chandra sheaves [7].

Theorem A.5. *Let* $\mathcal{I}(Q, \tau)$ *be a standard module in* $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \omega)$. *Then*

(i) $\mathcal{I}(Q, \tau)$ *has finite length;*

(ii) $\mathcal{I}(Q, \tau)$ *has a unique irreducible submodule* $\mathcal{L}(Q, \tau)$*;*

(iii) *if* $\mathcal{L}(Q', \tau')$ *is the irreducible submodule of another standard module* $\mathcal{I}(Q', \tau')$*, then*

$$\mathcal{L}(Q, \tau) \cong \mathcal{L}(Q', \tau') \Longleftrightarrow Q = Q' \text{ and } \tau \cong \tau'.$$

Theorem A.6. *Every irreducible object in* $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \omega)$ *is isomorphic to a unique* $\mathcal{L}(Q, \tau)$ *for some* K*-orbit* Q *in* X *and irreducible* K*-equivariant* $\mathcal{O}_Q$*-module* τ *compatible with* $(\lambda, \omega) \in \mathfrak{h}^* \times \mathfrak{k}^*$.

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