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K-THEORY DOESN'T EXIST

Ethan AKIN
The City College of New York, New York, U.S.A.

"Anyone can prove true theorems" — *W. Feller*

The common room in old Fine Hall at Princeton was dominated by a long oak table with a frayed, green linoleum top. It was the playing field for Dinkelspiel, a game which involved sliding old Go stones from the east end of the table toward the west with the object of getting a number of one's stones ahead of all of the opponent's. Dinkelspiel was played at all hours except tea-time when the table was rather inconveniently covered with cups. Between bouts of Dinkelspiel and other games the graduate students gathered in the common room to discuss their progress, to challege one another with problems, to teach and to learn. The social traditions of a school and the people who embody them are often as important for a student's education as formal lectures and for me, Princeton will always be that common room, Dinkelspiel and George Cooke, the reigning champion.

The common room problems varied quite a bit in depth. We spent a morning puzzling over whether a subspace of codimension one in a Hilbert space has to be closed until Michael O'Nan walked in and said "Take the kernel of a discontinuous linear functional". Norman Levitt arrived one day with the riddle: "What has six points and the weak homotopy type of S^2?" (See [2]). Then there was Frank Larkin's celebrated announcement that "The sheaf of germs of well-formed formulae on the long, long line is paracompact". Or perhaps it was "... is not paracompact". I wasn't there so I'm not sure.

George was in the center of all this with the quickest answers and the most ridiculous problems. But the best times for me were as we sat in the corner of the room and he taught me homotopy theory. Expertise in a subject sometimes amounts to the knowledge of where one can go quickly and where one must proceed with care. It often makes the expert impatient to go slowly through elementary definitions for the benefit of the nervous novice. George had vast patience for this kind of teaching. I recall a week of struggle with definitions and orientations for the Whitehead product to get the signs right in the Jacobi identity.

My own common room problem didn't reach the common room because I ran into George by the door of Fine Hall and we worked on it at the blackboard down the hall. I had been reading Steenrod's book [1] and when I got to page 38 I proved that K-theory didn't exist. So I went to look for George.

We will now prove that every vector bundle is stably trivial and thus that $K(B)=0$ for any space B.

Begin with a principal G bundle $\pi : P \to B$. If we pull π back over itself:

$$\begin{tikzcd} P \circ P \arrow[r, "p_2"] \arrow[d, "p_1"'] & P \arrow[d, "\pi"] \\ P \arrow[r, "\pi"'] & B \end{tikzcd}$$

then the pullback G bundle p_1 over P is trivial. In fact, the diagonal map $\Delta : P \to P \times P$ maps into

$$P \circ P = \{(x_1, x_2) \in P \times P : \ \pi x_1 = \pi x_2\}$$

and so gives a section of p_1. Thus the diagram:

$$\begin{tikzcd} P \times G \arrow[r, "\mu"] \arrow[d, "p_1"'] & P \arrow[d, "\pi"] \\ P \arrow[r, "\pi"'] & B \end{tikzcd}$$

is a pull-back diagram where p_1 is the projection and μ is the action of G on P.

If $G = \mathrm{GL}(N; \mathbf{R})$ and π is the principal G bundle for a vector bundle ξ, then the $G \times G$ bundle $\pi \circ p_1 : P \times G \to B$ represents the Whitney sum $\xi \oplus \xi$, i.e. the associated $\mathbf{R}^{2n}$ bundle is $\xi \oplus \xi$.

On the other hand, if $p_0 : B \times G \to B$ is the trivial G bundle then clearly the diagram:

$$\begin{tikzcd} P \times G \arrow[r, "\pi \times 1"] \arrow[d, "p_1"'] & B \times G \arrow[d, "p_0"] \\ P \arrow[r, "\pi"'] & B \end{tikzcd}$$

is a pullback diagram.

Thus if $G = \mathrm{GL}(n)$, π is the principal bundle for ξ and p_0 is the principal bundle for the trivial $\mathbf{R}^n$ bundle ε, then the $G \times G$ bundle $\pi \circ p_1$: $P \times G \to B$ represents the Whitney sum $\xi \oplus \varepsilon$.

Since the $G \times G$ bundle $\pi \circ p_1$: $P \times G \to B$ has both $\xi \oplus \xi$ and $\xi \oplus \varepsilon$ as associated $\mathbf{R}^{2n}$ bundles, we have $\xi \oplus \xi = \xi \oplus \varepsilon$. If we now add a complimentary bundle ξ' for ξ to both sides, we get that $\varepsilon_1 \oplus \xi = \varepsilon_2$ where ε_1, ε_2 are trivial. Thus, ξ is stably trivial and so represents 0 in K-theory.

Actually, K-theory does exist. George Cooke and Larry Smith found the error for me. I hope the reader will play with the puzzle for a bit before proceeding on to the solution.

It is, in fact, true that the $G\times G$ bundle $\pi\circ p_1 : P\times G \to B$ of the first diagram represents $\xi\oplus\xi$ and the $G\times G$ bundle $\pi\circ p_1 : P\times G\to B$ of the second diagram represents $\xi\oplus\varepsilon$. What we have in fact constructed is two different $G\times G$ bundles with the same total space, base space, group and map. They are nonetheless different because the $G\times G$ actions are different.

$G\times G$ acts on $P\circ P$ by $(x_1, x_2)\circ(g_1, g_2) = (x_1g_1, x_2g_2)$. The isomorphism $P\times G\to P\circ P$ is given by $(x, g)\to(x, xg)$. Thus, the $G\times G$ action on $P\times G$ in the first case is given by $(x, g)\circ(g_1, g_2) = (xg_1, g_1^{-1}gg_2)$.

The $G\times G$ action on $P\times G$ in the second case is the product action $(x, g)\circ(g_1, g_2) = (xg_1, gg_2)$.

The moral is that a principal bundle is not characterized by its projection map. Note that a trivial principal bundle is characterized by its projection map since a principal bundle is trivial iff it admits a section.

References

[1] N. Steenrod, The Topology of Fibre Bundles (Princeton Univ. Press, Princeton, NJ, 1951).

[2] McCord, C. Michael, Singular homology groups and homotopy groups of finite topological spaces, Duke Math. J. 33 (1966) 465–474.