Chapter 10

The Condom Problem

A well known puzzle among combinatorialists is the following: \( m \) men and \( n \) women, each with a different sexually transmitted disease, want to engage in all \( mn \) sexual encounters with no one catching anyone else's disease and with the minimal number of condoms being used. You are allowed to nest condoms and to turn them inside out, but once a surface becomes infected by a disease it stays infected for all time. This problem first appeared in print in the euphemistic form of doctors, patients, and surgical gloves in Martin Gardner's column in *Isaac Asimov's Science Fiction Magazine* [36, Gar1] (also[37]). The paper of Hajnal and Lovász [55] (see Section 10.1) uses the terminology of rabbits, radioactive plates, and membranes while the paper of Orlitzky and Shepp [93] (see Exercise 10.2) talks about computers and interfaces.

The cases of \( m = n = 2 \) and \( m = 2k + 1, \ n = 1 \) were the original formulations of the problem (the case \( m = 3, \ n = 1 \) appeared in [37, page 134]), and these examples will be described before moving on to the general case. First, one needs to set up notation to formulate the problem in mathematical terms.

**Notation:** The men will be denoted by \( M_i, M'_j \), and the women by \( W_i, W'_j \). The condoms will be denoted by \( C_i \) with \( \overrightarrow{C_i} \) or \( \overleftarrow{C_i} \) in order to distinguish the direction used (\( C_i \) if no confusion is possible). An encounter will therefore be of the form

\[
M_a \overrightarrow{C_i} \overleftarrow{C_j} \ldots \overrightarrow{C_k} W_b,
\]
or if the diseases will are denoted by $D_i$

$D_a \overrightarrow{C_i} \overrightarrow{C_j} \cdots \overrightarrow{C_k} D_b$,

since the diseases uniquely identify the $M_i$’s and $W_j$’s. If there is no confusion the $\overrightarrow{C_i}$’s may be dropped.

The case $m = n = 2$: Two condoms is the answer. Let $M_1, M_2$ be the men and $W_1, W_2$ be the women, and $C_1, C_2$ be the condoms. The solution consists of the sequence of encounters

(1) $M_1 \overrightarrow{C_1} \overrightarrow{C_2} W_1$
(2) $M_1 \overrightarrow{C_1} W_2$
(3) $M_2 \overrightarrow{C_2} W_1$
(4) $M_2 \overrightarrow{C_2} \overrightarrow{C_1} W_2$

To see that two is the minimum possible note that each condom provides two clean surfaces, and there are four people involved.

The case $m = 2k + 1, n = 1$: $k + 1$ condoms is the answer. Label the men $M_1, M_2, \ldots, M_k, M'_1, M'_2, \ldots, M'_k$ and $M$. The woman will be denoted by $W$, and the condoms by $C_1, C_2, \ldots, C_k$, and $C$.

The solution consists of the sequence

(1) $M_1 \overrightarrow{C_1} \overrightarrow{C} W$
(2) $M_2 \overrightarrow{C_2} \overrightarrow{C} W$

\vdots

(k) $M_k \overrightarrow{C_k} \overrightarrow{C} W$

(k + 1) $M \overrightarrow{C} W$

(k + 2) $M'_1 \overrightarrow{C_1} \overrightarrow{C} W$

(k + 3) $M'_2 \overrightarrow{C_2} \overrightarrow{C} W$

\vdots

(2k + 2) $M'_k \overrightarrow{C_k} \overrightarrow{C} W$
10.1. THE WORK OF HAJNAL AND LOVÁSZ

As before it can be seen that \( k + 1 \) is optimal since this number is exactly equal to half the number of participants.

In their paper, A. Hajnal and L. Lovász [55] considered the condom problem (in an equivalent formulation) and they gave an elegant solution by providing upper and lower bounds that differed by an additive constant of one. I complete this analysis by presenting an algorithm that is optimal for all \( m \) and \( n \). I will show

**Theorem.** Assume there are \( m \) men and \( n \) women with \( m \geq n \) (if \( n > m \) then the proof is identical but with the role of men and women interchanged). Then there is an algorithm that uses

\[
\left\lfloor \frac{m}{2} + \frac{2n}{3} \right\rfloor
\]

condoms, where \( [x] \) represents the smallest integer greater than or equal to \( x \). (10.1) is optimal except for the case \( m = n = 2 \) when 2 condoms suffice, and the case \( n = 1, m = 2k + 1 \) when \( (m + 1)/2 \) condoms suffice (these are also optimal).

(Hajnal and Lovász provided a lower bound of \( \left\lfloor m/2 + 2n/3 - 1/3 \right\rfloor \) for all \( m, n \) and an upper bound of \( \left\lfloor m/2 + 2n/3 \right\rfloor + 1 \) in the case of \( m = n = 6k \).)

10.1 The work of Hajnal and Lovász

10.1.1 The algorithm

The Hajnal and Lovász algorithm works for \( m = 2k \) men and and \( n = 3\ell \) women, where \( n \geq m \), and uses \( n/2 + 2m/3 + 1 \) condoms. The method is identical to one with 2 men and 3 women and 4 condoms so this will be used to illustrate the algorithm, see Figure 10.1

**Note on the figures:**

- denotes a person not protected by a condom.

- denotes a person protected by a condom with a clean other side.
\(\overset{\circ}{\top}\) denotes a person protected by a condom with an infected other side.

* denotes a person who has finished all encounters.

Labeling the men \(M_1, M_2\) and the women \(W_1, W_2, W_3\), and the condoms \(C_1, C_2, C_3, J\) one has the algorithm (following Figure 10.1)

**Algorithm for two men, three women, four condoms:**

(a) \(M_1\) will use \(C_1\), \(W_1\) will use \(C_2\), \(W_2\) will use \(C_3\). \(J\) will be the *master condom* which always has a clean side and is used as an accessory to protect \(C_i\)’s that need to stay clean.

(b) The encounters \(M_1C_1C_2W_1, M_1C_1C_3W_2\).

(c) \(W_2\) gives up \(C_2\) which is turned inside out and will be used by \(W_3\) for all time.

(d) Encounter \(M_1C_1J\overset{\circ}{\top}C_2W_3\).

(e) \(M_1\) has completed all encounters and so gives up \(C_1\) which is turned inside out and will be used by \(M_2\) for all time.

(f) Encounters \(M_2C_1J\overset{\circ}{\top}C_2W_1, M_1C_1C_2W_3\).

(g) \(W_1\) has completed all encounters, and \(C_2\) is turned inside out and used by \(W_2\).

(h) Encounter \(M_2C_1C_2W_2\).

The method for \(m = 2k\) men, \(n = 3\ell\) women, and \(k + 2\ell + 1 = m/2 + 2n/3 + 1\) condoms, where \(m \geq n\), is very similar to the above algorithm. Label the men \(M_1, \ldots, M_m\), and the women \(W_1, \ldots, W_n\), and the condoms \(C_1, \ldots, C_k, C'_1, \ldots, C'_{2\ell}\). Then the algorithm is as above but replacing all encounters of the form \(M_i \leftrightarrow W_j\) in the above with \(M_{2a+i} \leftrightarrow W_{3b+j}, a = 1, \ldots, k, b = 1, \ldots, \ell\).
Figure 10.1: Two men, three women, four condoms.
Algorithm for $n$ men, $m$ women, $n/2 + 2m/3 + 1$ condoms:

(a) $M_{2i-1}$ will use condom $C_i$, $i = 1, \ldots, k$. $W_{3j-2}$ will use $C'_{2j-1}$. $W_{3j-1}$ will use $C'_{2j}$, $j = 1, \ldots, \ell$. $J$ will be the master condom which always has a clean side and is used as an accessory to protect $C_i$'s that need to stay clean.

(b) All encounters $M_{2i-1}C_iC'_{2j-1}W_{3j-2}$, $M_{2i-1}C_iC'_{2j}W_{3j-1}$.

(c) Each $C'_{2j}$ is turned inside out and will be used by $W_{3j}$ for all time.

(d) All encounters $M_{2i-1}C_i \overrightarrow{J} C'_{2j}W_{3j}$.

(e) Each $M_{2i-1}$ has completed all encounters and so each $C_i$ is turned inside out and will be used by $M_{2i}$ for all time.

(f) All encounters $M_{2i}C_i \overleftarrow{J} C'_{2j-1}W_{3j-2}$, $M_{2i}C_iC'_{2j}W_{3j}$.

(g) Each $W_{3j-2}$ has completed all encounters so gives up $C'_{2j-1}$ which is turned inside out and used by $W_{2j-1}$.

(h) All encounters $M_{2i}C_iC'_{2j-1}W_{2j-1}$.

Remark: This gives that $6n$ men and $6n$ women only need $7n + 1$ condoms.

10.1.2 The lower bound

As above, let $M_1, M_2, \ldots, M_m$ be the men and $W_1, W_2, \ldots, W_n$ be the women. Following [55] these will be the vertices of a graph. The condoms will be edges of this graph.

A surface of a condom $C$ becomes infected by a disease if the surface was clean and then came into contact with this disease only.

This is how the graph is connected:

1. If a side of a condom $C$ becomes infected first with the disease corresponding to person $D_1$ and then the other side becomes infected with the disease corresponding to person $D_2$ then we draw a directed edge $D_1 \rightarrow D_2$ connecting person $D_1$ to $D_2$. 
2. If the infection occurred simultaneously then an arbitrary direction is chosen. If a side of condom becomes infected by two diseases simultaneously then we connect this side to an arbitrary vertex, likewise if the side remains clean throughout.

For example the reader can check that the solution for $m = n = 2$ given in the introduction has the graph

```
  • → •
  ↑   ↓
```

and the solution for $n = 1, m = 2k + 1$ has the graph

```
  • ← (m−1)/2
  • ← •
  ↑
```

The Hajnal and Lovász algorithm given above has the graph

```
  • ← • ← •
  m/2 times

  • ← •
  2n/3 times
```

The next step is to find a lower bound, but first here is a general observation about the optimal way to use condoms. The idea is to turn a condom inside out and pass it to someone after you’ve used it. For example the most efficient way is for one person $D_1$ to use a condom for all interactions, then turn it inside out and give it to person $D_2$ who then uses it for all interactions. This gives the connected two-component

\[(10.2) \quad D_1 \rightarrow D_2.\]

The next most efficient method is for three people to use two condoms

\[D_1 \rightarrow D_2 \rightarrow D_3,\]
where \( D_2 \) passes to \( D_3 \) and then \( D_1 \) passes to \( D_2 \), and so forth. The Hajnal-Lovász algorithm chooses to use 2-components for either the men or the women, depending, on which are more numerous, and then uses 3-components for the others.

The next observation is that in (10.2), \( D_i \) must accomplish all encounters before passing to \( D_j \) and so \( D_j \) must wait for this before beginning. This immediately implies:

(i) two connected components of the form

\[
M_i \leftrightarrow M_j \quad W_k \leftrightarrow W_\ell
\]

cannot occur. In other words, only the men or only the women can contain a 2-component.

(ii) If two connected components

\[
M_i \leftrightarrow W_j \quad M_k \leftrightarrow W_\ell
\]

exist, then one must point toward the woman and the other toward the man.

Now using the observation that no vertex in the graph is isolated and the restrictions of (i) and (ii) it follows that an optimal solution must look something like

\[
\begin{array}{c}
\text{(m-2)/2 times}
\end{array}
\]

\[
\begin{array}{c}
\text{rest of women}
\end{array}
\]

where the "rest of the women" part of the graph does not contain a 2-component. It follows there must be at least

\[
\frac{m-2}{2} + 2 + \frac{2}{3} (n-2) = \frac{m}{2} + \frac{2n}{3} - \frac{1}{3}
\]

edges and so at least \( \lceil m/2 + 2n/3 - 1/3 \rceil \) condoms.

**Remark:** In the case \( 6n \) men and \( 6n \) women, one sees that the lower bound gives \( 7n \) condoms, so the upper and lower bounds of Hajnal and Lovász differ by one. This upper bound will be reduced in the next section.
10.2 The algorithm

Before doing the general case, I'll describe the algorithm for 4 men, 6 women, and 6 condoms since this case already contains all the ingredients of the general case. The key idea is to fabricate the master condom from the ones used by the men and women, instead of just adding an extra one as in the Hajnal-Lovász algorithm.

As before, label the men $M_1, \ldots, M_4$, and the women $W_1, \ldots, W_6$. Here is a description of the algorithm:

Algorithm for four men, six women, six condoms:

(a) $M_2$ will use condom $C_1$, $M_3$ will use $C_2$, $W_1$ will use $C_3$, and $W_3, W_4, W_5$ will use $C_4, C_5, C_6$ respectively.

(b) All encounters $M_2C_1$ and $M_3C_2$ with $C_3W_1$, $C_4W_3$, $C_5W_4$, and $C_6W_5$.

(c) $W_4$ gives up $C_5$ which will then be used as a master condom, and so will be denoted by $J$ for the rest of the algorithm. $W_5$ passes $C_6$ to $W_6$. 
(d) Encounters $M_2C_1 \overrightarrow{J} C_6W_6$ and $M_3C_2 \overrightarrow{J} C_6W_6$.

(e) $M_2$ has completed all encounters except $W_2$, so will no longer need $C_1$. One therefore has the encounter $M_2C_1W_2$.

(f) $M_2$ now passes $C_1$ to $W_2$.

(g) Encounter $M_3C_2 \overrightarrow{J} C_1W_2$. 
(h) $M_3$ has completed all encounters so passes $C_2$ to $M_4$.

(i) Encounter $M_4C_2J\bigvee C_3W_1$

(j) $W_1$ has completed all encounters except with $M_1$ so that one has the encounter $M_1C_3W_1$.

(k) $W_1$ passes $C_3$ to $M_1$. 
(1) Encounters $M_1C_3$ and $M_4C_2$ with $C_1W_2$ and $C_6W_6$.

![Diagram](image1)

(m) Encounters $M_1C_3$ and $M_4C_2$ with $\overline{J}C_4W_3$.

![Diagram](image2)

(n) $W_3$ has finished so passes $C_4$ to $W_4$. $W_5$ uses $J$ from now on.

![Diagram](image3)

(o) Encounters $M_1C_3$ and $M_4C_2$ with $C_4W_4$ and $JW_5$.

![Diagram](image4)

The point of this much more complicated method is that the master condom was generated by using a four-component in a nontrivial way. Note that this method needs to use two connected components that go from a man to a woman and from a woman to a man.

We now proceed to the general case of $m = 2k+2$ men and $n = 3\ell+6$ women, $m/2 + 2n/3 = k + 2\ell + 5$ condoms, where $m \geq n$. This is very
similar to the previous algorithm. One labels the men $M_1, M_2,$ and $M'_i, i = 1, 2, \ldots, 2k,$ and labels the women $W_1, W_2, W_3, W_4, W_5, W_6,$ and $W'_j, j = 1, \ldots, 3\ell.$ Also label the condoms $C_1, C_2, C_3, C_4, C_5, C'_i, i = 1, \ldots, k, C''_j, j = 1, \ldots, 2\ell.$ The steps of the algorithm are:

**Algorithm for $m$ men, $n$ women, $m/2 + 2n/3$ condoms:**

(a) $M_2$ will use condom $C_1,$ $M'_{2i-1}$ will use $C'_i, i = 1, \ldots, k.$ $W_1$ will use $C_2,$ and $W_3, W_4, W_5$ will use $C_3, C_4, C_5$ respectively. $W'_{3j-2}$ will use $C''_{2j-1}, j = 1, \ldots, \ell$ and $W'_{3j-1}$ will use $C''_{2j}, j = 1, \ldots, \ell.$

(b) All encounters $M_2 C_1$ and $M'_{2i-1} C'_i$ with $C_2 W_1, C_3 W_3, C_4 W_4, C_5 W_5, C''_{2j-1} W'_{3j-2}, C''_{2j} W'_{3j-1}.$

(c) $W_4$ gives up $C_4$ which will then be used as a master condom, and so will be denoted by $J$ for the rest of the algorithm. $W_5$ passes $C_5$ to $W_6.$ Each $W'_{3j-1}$ passes $C''_{2j}$ to $W'_{2j}.$
(d) Encounter $M_2 C_1 \rightarrow J C_5 W_6$, all encounters $M_2 C_1 \rightarrow J C_2 W_3$. All encounters $M_{2i-1} C_i \rightarrow J C_5 W_6$ and all encounters $M_{2i-1} C_i \rightarrow J C_2 W_3$.

(e) $M_2$ has completed all encounters except $W_2$, so will no longer need $C_1$. One therefore has the encounter $M_2 C_1 W_2$.

(f) $M_2$ now passes $C_1$ to $W_2$. 
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(g) All encounters $M'_{2i-1} C_i' \overline{J} C_1 W_2$.

(h) Each $M'_{2i-1}$ has completed all encounters so passes $C_i'$ to $M'_{2i}$.

(i) All encounters $M'_{2i} C_i' \overline{J} C_2 W_1$

(j) $W_1$ has completed all encounters except with $M_1$ so that one has the encounter $M_1 C_2 W_1$. 
(k) $W_1$ passes $C_2$ to $M_1$.

(1) Encounters $M_1C_2$ and $M_2'C_i$ with $C_1W_2$, $C_5W_6$, and all $C_{2j}W_{3j}$.

(m) Encounters $M_1C_2$ and $M_2'C_i$ with $\overleftarrow{J}C_3W_3$ and all $C''_{2j-1}W'_{3j-2}$.

(n) $W_3$ has finished so passes $C_3$ to $W_4$. $W_5$ uses $J$ from now on. Each $W'_{3j-2}$ has finished and passes $C''_{2j-1}$ to $W'_{3j-1}$.
10.3. THE GENERAL CASE

(o) All remaining encounters $M_1C_2$ and $M'_2C'_i$ with $C_3W_4$, $JW_5$, and all $C'_{2j-1}W_{3j-1}$.

**Remark:** In the case of $6n$ men and $6n$ women this algorithm needs $6n/2 + 2 \cdot 6n/3 = 7n$ condoms. By the lower bound of Section 10.1.2, this algorithm is optimal.

Here is a graph of the algorithm

10.3 The general case

The algorithm of the last section can be modified to treat all the cases for $m, n \geq 6$. We only present the generic cases since the others can be handled similarly.

(a) $m \equiv 1 \pmod{2}$ and $n \equiv 2 \pmod{3}$

(b) $m \equiv 0 \pmod{2}$ and $n \equiv 2 \pmod{3}$

(a) We have an algorithm similar to the above based on the graph
(b) This case uses a different algorithm (similar to the Hajnal-Lovász method) using the graph

\[
\begin{array}{c}
\bullet & \bullet \\
\uparrow & \downarrow \\
\bullet & \bullet
\end{array} \rightarrow \quad \begin{array}{c}
\bullet \\
\rightarrow \quad \bullet
\end{array} \quad J = \text{master condom}
\]

where \( J \) serves as a "master condom," i.e., one side is always clean and the other always unclean, as before, but it never comes in contact directly with a \( M_i \) or \( W_j \).

As in \([55]\) I leave the case of \( n < 6 \) to the reader. Note, however, that the cases \( m = n = 2 \) and \( n = 1, m = 2k + 1 \), are the only ones for which (10.1) can be improved.

### 10.4 Lower bounds

To show that (10.1) is optimal, we must improve the lower bound of (10.3).

**Lemma 1.** If \( m \geq n \geq 6 \), then (10.1) is optimal.

(As before, I leave the case \( n < 6 \) to the reader.)

**Proof:** The only case when the lower bound of (1) doesn't follow directly from (10.3) is if

\[
\left\lfloor \frac{m}{2} + \frac{2n}{3} \right\rfloor \neq \left\lfloor \frac{m}{2} + \frac{2n}{3} - \frac{1}{3} \right\rfloor
\]

which happens when

(a) \( m \equiv 0 \pmod{2} \) and \( n \equiv 2 \pmod{3} \)

(b) \( m \equiv 1 \pmod{2} \) and \( n \equiv 1 \pmod{3} \)

We will prove the Lemma for (a). The proof for (b) is similar.
Proof for (a): By the comments at the end of §2 the graph of an algorithm taking \( \frac{m}{2} + \frac{2n}{3} - \frac{1}{3} \) condoms must look like

\[
\begin{array}{c}
\bullet \quad \bullet \\
\uparrow \quad \downarrow \\
\bullet \quad \bullet
\end{array}
\]

\[
\begin{array}{c}
\bullet \quad \bullet \quad \rightarrow \quad \bullet \\
\bullet \quad \bullet \quad \leftarrow \quad \bullet \quad \leftarrow \quad \bullet
\end{array}
\]

\[(10.4)\]

I leave the proof for \( n = 2 \) to the reader. For \( n > 2 \) I will need the fact that the 3-components in (10.4) must all be of the form \( \bullet \rightarrow \bullet \rightarrow \bullet \). As a matter of fact, the following stronger statement is true:

**Lemma 2.** For any algorithm, if the men have a 2-component, then the 3-components of the women must be of the form \( \bullet \rightarrow \bullet \rightarrow \bullet \).

For the moment assume that Lemma 2 is true, then the graph (10.4) looks like

\[
\begin{array}{c}
\bullet \quad \bullet \quad \rightarrow \quad \bullet \\
\bullet \quad \bullet \quad \leftarrow \quad \bullet \quad \leftarrow \quad \bullet
\end{array}
\]

Now consider the last \( W_{3i+5}M_3 \) for \( 0 \leq i \leq (n-5)/3 \), say it is \( W_{3j+6}M_3 \). \( M_3 \) has to be protected, but I will show that no intermediate condoms can be used: It is fairly clear that \( W_1M_1 \) and \( M_2W_2 \) can't be used. None of the \( W_{3i+3}W_{3i+4} \) can be used since one side is reserved for \( W_{3i+3}M_4 \) and the other is reserved for \( W_{3i+4}M_4 \). Finally, by assumption all other \( W_{3i+4}W_{3i+4} \) are unclean on both sides. Therefore \( W_{3j+5}M_3 \) is impossible.

**Proof of Lemma 2:** Let \( M_1 \rightarrow M_2 \) be a 2-component of the men, and \( W_1 \leftarrow W_2 \leftarrow W_3 \) be a 3-component of the women. The possibilities for the 3-component that have to be ruled out are:

(i) \( \bullet \leftarrow \bullet \rightarrow \bullet \)

(ii) \( \bullet \rightarrow \bullet \leftarrow \bullet \)

We consider the case of (i) (the (ii) case is handled similarly).

We try to figure out in which order \( W_1, W_2, W_3 \) will have sex with \( M_2 \). Without loss of generality, one can assume that \( W_2W_3 \) will be used
for \( W_2M_2 \). Now by definition of the directed graph \( \overrightarrow{M_1M_2} \) is unclean on both sides. Furthermore, since \( M_1 \) has completed all its encounters, \( \overrightarrow{W_2W_1} \) and \( \overrightarrow{W_2W_3} \) are unclean on both sides (this follows from the shape of the component).

(a) \( W_2 \) cannot be first since this would mean that the \( W_3 \) side of \( \overrightarrow{W_2W_3} \) would have to be protected, which is impossible, since a new condom would become infected by \( W_3 \), and \( W_3 \) only has one edge incident to it.

(b) \( W_3 \) cannot be first since the \( W_2 \) side of \( \overrightarrow{W_2W_3} \) would have to be protected. From the assumption about the 3-component, this means that \( \overrightarrow{W_2W_1} \) would have to be put over \( \overrightarrow{W_2W_3} \). But the \( W_1 \) side of \( \overrightarrow{W_2W_1} \) would have to be protected, which is impossible as before.

(c) Using the same argument as in (b) it is seen that \( W_1 \) cannot be first.

This means that none of \( W_1, W_2, W_3 \) can be first and the lemma follows by contradiction.

**Exercise 10.1** Show that a formula similar to (10.1) holds for the case where all pairs of individuals have a sexual encounter.

**Exercise 10.2** [M] (A. Orlitzky and L. Shepp [93]) Analyze the condom problem with \( m \) men, \( n \) women, \( m > n \) where all men are bisexual.

**Exercise 10.3** [M] In general consider the condom problem where a preference graph is given and two people have a sexual encounter if and only if they are joined by an edge (thus the original problem corresponds to the complete bipartite graph).