Chapter 10

The Condom Problem

A well known puzzle among combinatorialists is the following: m men and n women, each with a different sexually transmitted disease, want to engage in all mn sexual encounters with no one catching anyone else's disease and with the minimal number of condoms being used. You are allowed to nest condoms and to turn them inside out, but once a surface becomes infected by a disease it stays infected for all time. This problem first appeared in print in the euphemistic form of doctors, patients, and surgical gloves in Martin Gardner's column in *Isaac Asimov's Science Fiction Magazine* [36, Gar1] (also[37]). The paper of Hajnal and Lovász [55] (see Section 10.1) uses the terminology of rabbits, radioactive plates, and membranes while the paper of Orlitzky and Shepp [93] (see Exercise 10.2) talks about computers and interfaces.

The cases of $m = n = 2$ and $m = 2k + 1$, $n = 1$ were the original formulations of the problem (the case $m = 3$, $n = 1$ appeared in [37, page 134]), and these examples will be described before moving on to the general case. First, one needs to set up notation to formulate the problem in mathematical terms.

Notation: The men will be denoted by M_i, M'_j, and the women by W_i, W'_j. The condoms will be denoted by C_i with $\overrightarrow{C_i}$ or $\overleftarrow{C_i}$ in order to distinguish the direction used (C_i if no confusion is possible). An encounter will therefore be of the form

$$M_a \overrightarrow{C_i}\ \overrightarrow{C_j} \cdots \overleftarrow{C_k} W_b ,$$

or if the diseases will are denoted by D_i

$$D_a \overrightarrow{C_i}\ \overrightarrow{C_j} \cdots \overleftarrow{C_k}\, D_b\,,$$

since the diseases uniquely identify the M_i's and W_j's. If there is no confusion the $\overrightarrow{C}$'s may be dropped.

The case $m = n = 2$: Two condoms is the answer. Let M_1, M_2 be the men and W_1, W_2 be the women, and C_1, C_2 be the condoms. The solution consists of the sequence of encounters

(1) $$M_1 \overrightarrow{C_1}\ \overrightarrow{C_2}\, W_1$$

(2) $$M_1 \overrightarrow{C_1}\, W_2$$

(3) $$M_2 \overrightarrow{C_2}\, W_1$$

(4) $$M_2 \overrightarrow{C_2}\ \overrightarrow{C_1}\, W_2$$

To see that two is the minimum possible note that each condom provides two clean surfaces, and there are four people involved.

The case $m = 2k+1$, $n = 1$: $k+1$ condoms is the answer. Label the men $M_1, M_2, ..., M_k$, $M'_1, M'_2, \ldots, M'_k$ and M. The woman will be denoted by W, and the condoms by $C_1, C_2, \ldots, C_k$, and C.

The solution consists of the sequence

(1) $$M_1 \overrightarrow{C_1}\ \overrightarrow{C}\, W$$

(2) $$M_2 \overrightarrow{C_2}\ \overrightarrow{C}\, W$$

$$\vdots$$

(k) $$M_k \overrightarrow{C_k}\ \overrightarrow{C}\, W$$

($k+1$) $$M \overrightarrow{C}\, W$$

($k+2$) $$M'_1 \overleftarrow{C_1}\ \overrightarrow{C}\, W$$

($k+3$) $$M'_2 \overleftarrow{C_2}\ \overrightarrow{C}\, W$$

$$\vdots$$

($2k+2$) $$M'_k \overleftarrow{C_k}\ \overrightarrow{C}\, W$$

As before it can be seen that $k+1$ is optimal since this number is exactly equal to half the number of participants.

In their paper, A. Hajnal and L. Lovász [55] considered the condom problem (in an equivalent formulation) and they gave an elegant solution by providing upper and lower bounds that differed by an additive constant of one. I complete this analysis by presenting an algorithm that is optimal for all m and n. I will show

Theorem. *Assume there are m men and n women with $m \geq n$ (if $n > m$ then the proof is identical but with the role of men and women interchanged). Then there is an algorithm that uses*

$$\left\lceil \frac{m}{2} + \frac{2n}{3} \right\rceil \tag{10.1}$$

condoms, where $\lceil x \rceil$ represents the smallest integer greater than or equal to x. (10.1) is optimal except for the case $m = n = 2$ when 2 condoms suffice, and the case $n = 1$, $m = 2k+1$ when $(m+1)/2$ condoms suffice (these are also optimal).

(Hajnal and Lovász provided a lower bound of $\lceil m/2 + 2n/3 - 1/3 \rceil$ for all m, n and an upper bound of $\lceil m/2 + 2n/3 \rceil + 1$ in the case of $m = n = 6k$.)

10.1 The work of Hajnal and Lovász

10.1.1 The algorithm

The Hajnal and Lovász algorithm works for $m = 2k$ men and and $n = 3\ell$ women, where $n \geq m$, and uses $n/2 + 2m/3 + 1$ condoms. The method is identical to one with 2 men and 3 women and 4 condoms so this will be used to illustrate the algorithm, see Figure 10.1

Note on the figures:

- $\bullet$ denotes a person not protected by a condom.

$\overline{\bullet}$ denotes a person protected by a condom with a clean other side.

$\frac{\uparrow}{\bullet}$ denotes a person protected by a condom with an infected other side.

$*$ denotes a person who has finished all encounters.

Labeling the men M_1, M_2 and the women W_1, W_2, W_3, and the condoms C_1, C_2, C_3, J one has the algorithm (following Figure 10.1)

Algorithm for two men, three women, four condoms:

(a) M_1 will use C_1, W_1 will use C_2, W_2 will use C_3. J will be the *master condom* which always has a clean side and is used as an accessory to protect C_i's that need to stay clean.

(b) The encounters $M_1C_1C_2W_1$, $M_1C_1C_3W_2$.

(c) W_2 gives up C_2 which is turned inside out and will be used by W_3 for all time.

(d) Encounter $M_1C_1\overrightarrow{J}C_2W_3$.

(e) M_1 has completed all encounters and so gives up C_1 which is turned inside out and will be used by M_2 for all time.

(f) Encounters $M_2C_1\overleftarrow{J}C_2W_1$, $M_1C_1C_2W_3$.

(g) W_1 has completed all encounters, and C_2 is turned inside out and used by W_2.

(h) Encounter $M_2C_1C_2W_2$.

The method for $m = 2k$ men, $n = 3\ell$ women, and $k + 2\ell + 1 = m/2 + 2n/3 + 1$ condoms, where $m \geq n$, is very similar to the above algorithm. Label the men $M_1, \ldots, M_m$, and the women $W_1, \ldots, W_n$, and the condoms $C_1, \ldots, C_k$, $C'_1, \ldots, C'_{2\ell}$. Then the algorithm is as above but replacing all encounters of the form $M_i \Leftrightarrow W_j$ in the above with $M_{2a+i} \Leftrightarrow W_{3b+j}$, $a = 1, \ldots, k$, $b = 1, \ldots, \ell$.

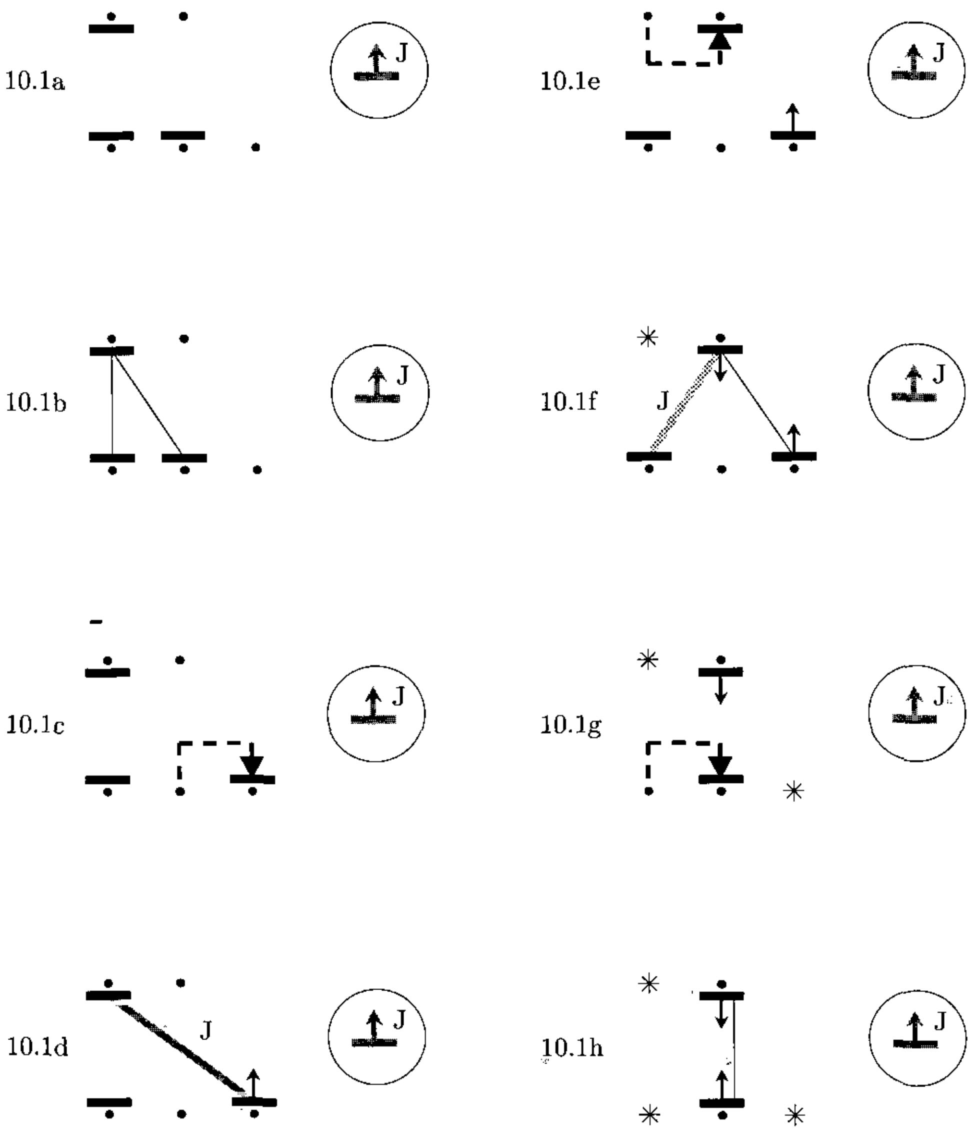


Figure 10.1: Two men, three women, four condoms.

Algorithm for n men, m women, $n/2 + 2m/3 + 1$ condoms:

(a) M_{2i-1} will use condom C_i, $i = 1, \ldots, k$. W_{3j-2} will use C'_{2j-1}, W_{3j-1} will use C'_{2j}, $j = 1, \ldots, \ell$. J will be the *master condom* which always has a clean side and is used as an accessory to protect C_i's that need to stay clean.

(b) All encounters $M_{2i-1}C_iC'_{2j-1}W_{3j-2}$, $M_{2i-1}C_iC'_{2j}W_{3j-1}$.

(c) Each C'_{2j} is turned inside out and will be used by W_{3j} for all time.

(d) All encounters $M_{2i-1}C_i\overrightarrow{J}C'_{2j}W_{3j}$.

(e) Each M_{2i-1} has completed all encounters and so each C_i is turned inside out and will be used by M_{2i} for all time.

(f) All encounters $M_{2i}C_i\overleftarrow{J}C'_{2j-1}W_{3j-2}$, $M_{2i}C_iC'_{2j}W_{3j}$.

(g) Each W_{3j-2} has completed all encounters so gives up C'_{2j-1} which is turned inside out and used by W_{2j-1}.

(h) All encounters $M_{2i}C_iC'_{2j-1}W_{2j-1}$.

Remark: This gives that $6n$ men and $6n$ women only need $7n + 1$ condoms.

10.1.2 The lower bound

As above, let $M_1, M_2, \ldots, M_m$ be the men and $W_1, W_2, \ldots, W_n$ be the women. Following [55] these will be the vertices of a graph. The condoms will be edges of this graph.

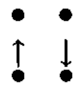
A surface of a condom C becomes *infected* by a disease if the surface was clean and then came into contact with this disease only.

This is how the graph is connected:

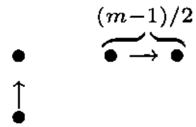
1. If a side of a condom C becomes infected first with the disease corresponding to person D_1 and then the other side becomes infected with the disease corresponding to person D_2 then we draw a directed edge $\overrightarrow{D_1 D_2}$ connecting person D_1 to D_2.

2. If the infection occurred simultaneously then an arbitrary direction is chosen. If a side of condom becomes infected by two diseases simultaneously then we connect this side to an arbitrary vertex, likewise if the side remains clean throughout.

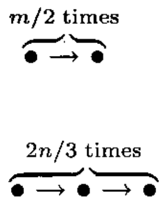
For example the reader can check that the solution for $m = n = 2$ given in the introduction has the graph



and the solution for $n = 1$, $m = 2k + 1$ has the graph



The Hajnal and Lovász algorithm given above has the graph



The next step is to find a lower bound, but first here is a general observation about the optimal way to use condoms. The idea is to turn a condom inside out and pass it to someone after you've used it. For example the most efficient way is for one person D_1 to use a condom for all interactions, then turn it inside out and give it to person D_2 who then uses it for all interactions. This gives the connected two-component

$$D_1 \to D_2 \,. \tag{10.2}$$

The next most efficient method is for three people to use two condoms

$$D_1 \to D_2 \to D_3 \,,$$

where D_2 passes to D_3 and then D_1 passes to D_2, and so forth. The Hajnal-Lovász algorithm chooses to use 2-components for either the men or the women, depending, on which are more numerous, and then uses 3-components for the others.

The next observation is that in (10.2), D_i must accomplish all encounters before passing to D_j and so D_j must wait for this before beginning. This immediately implies:

(i) two connected components of the form

$$M_i \longleftrightarrow M_j \qquad W_k \longleftrightarrow W_\ell$$

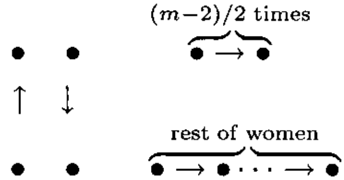
cannot occur. In other words, only the men or only the women can contain a 2-component.

(ii) If two connected components

$$M_i \longleftrightarrow W_j \qquad M_k \longleftrightarrow W_\ell$$

exist, then one must point toward the woman and the other toward the man.

Now using the observation that no vertex in the graph is isolated and the restrictions of (i) and (ii) it follows that an optimal solution must look something like



where the "rest of the women" part of the graph does not contain a 2-component. It follows there must be at least

$$(10.3) \qquad \frac{m-2}{2} + 2 + \frac{2}{3}(n-2) = \frac{m}{2} + \frac{2n}{3} - \frac{1}{3}$$

edges and so at least $\lceil m/2 + 2n/3 - 1/3 \rceil$ condoms.

Remark: In the case $6n$ men and $6n$ women, one sees that the lower bound gives $7n$ condoms, so the upper and lower bounds of Hajnal and Lovász differ by one. This upper bound will be reduced in the next section.

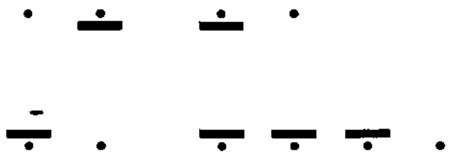
10.2 The algorithm

Before doing the general case, I'll describe the algorithm for 4 men, 6 women, and 6 condoms since this case already contains all the ingredients of the general case. The key idea is to fabricate the master condom from the ones used by the men and women, instead of just adding an extra one as in the Hajnal-Lovász algorithm.

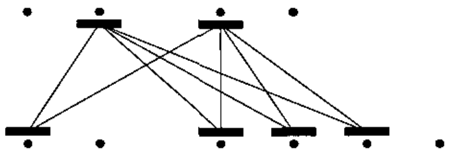
As before, label the men $M_1, \ldots, M_4$, and the women $W_1, \ldots, W_6$. Here is a description of the algorithm

Algorithm for four men, six women, six condoms:

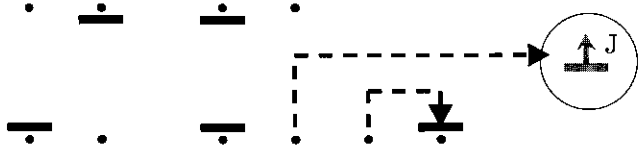
(a) M_2 will use condom C_1, M_3 will use C_2, W_1 will use C_3, and W_3, W_4, W_5 will use C_4, C_5, C_6 respectively.



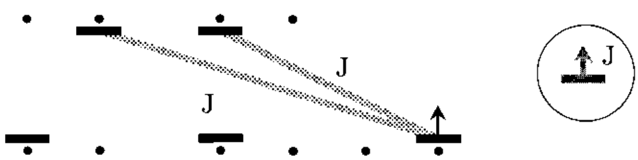
(b) All encounters M_2C_1 and M_3C_2 with C_3W_1, C_4W_3, C_5W_4, and C_6W_5.



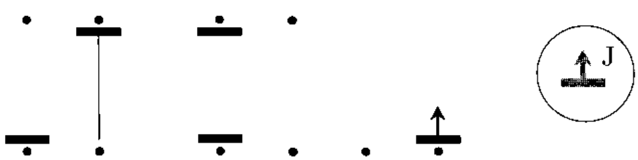
(c) W_4 gives up C_5 which will then be used as a master condom, and so will be denoted by J for the rest of the algorithm. W_5 passes C_6 to W_6.



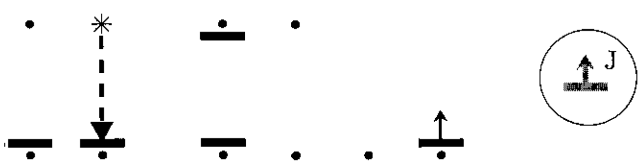
(d) Encounters $M_2C_1\overrightarrow{J}C_6W_6$ and $M_3C_2\overrightarrow{J}C_6W_6$.



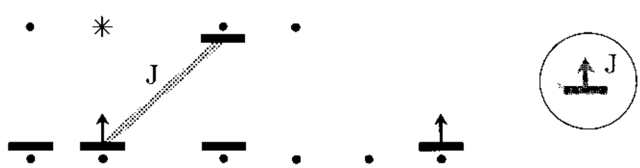
(e) M_2 has completed all encounters except W_2, so will no longer need C_1. One therefore has the encounter $M_2C_1W_2$.



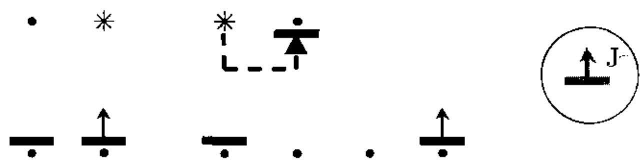
(f) M_2 now passes C_1 to W_2.



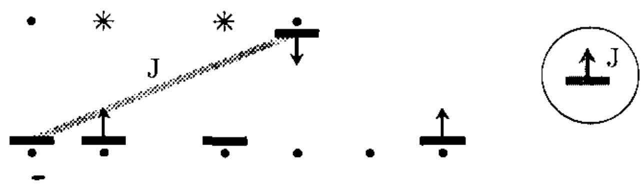
(g) Encounter $M_3C_2\overrightarrow{J}C_1W_2$.



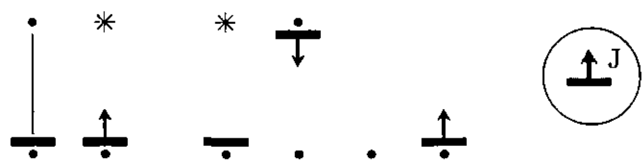
(h) M_3 has completed all encounters so passes C_2 to M_4.



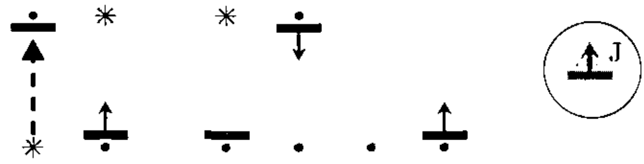
(i) Encounter $M_4C_2\overleftarrow{J}\,C_3W_1$



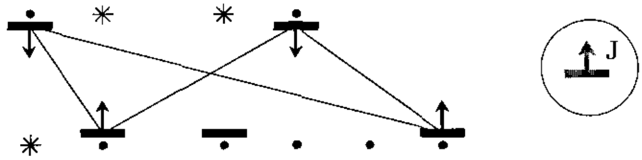
(j) W_1 has completed all encounters except with M_1 so that one has the encounter $M_1C_3W_1$.



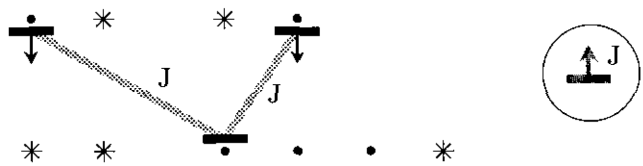
(k) W_1 passes C_3 to M_1.



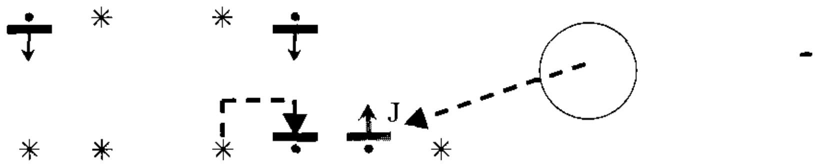
(**l**) Encounters M_1C_3 and M_4C_2 with C_1W_2 and C_6W_6.



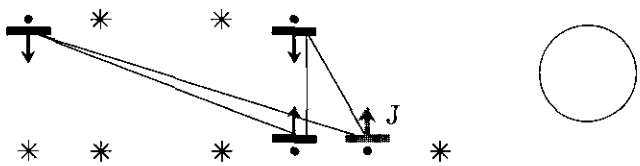
(**m**) Encounters M_1C_3 and M_4C_2 with $\overleftarrow{J}\,C_4W_3$.



(**n**) W_3 has finished so passes C_4 to W_4. W_5 uses J from now on.



(**o**) Encounters M_1C_3 and M_4C_2 with C_4W_4 and JW_5.

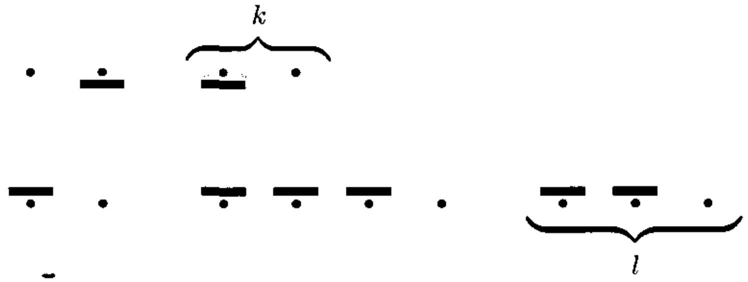


The point of this much more complicated method is that the master condom was generated by using a four-component in a nontrivial way. Note that this method needs to use two connected components that go from a man to a woman and from a woman to a man.

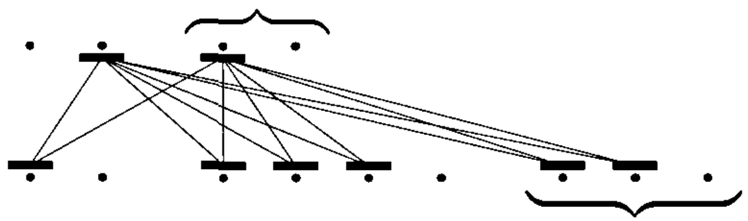
We now proceed to the general case of $m = 2k{+}2$ men and $n = 3\ell{+}6$ women, $m/2 + 2n/3 = k + 2\ell + 5$ condoms, where $m \geq n$. This is very similar to the previous algorithm. One labels the men M_1, M_2, and M'_i, $i = 1, 2, \ldots, 2k$, and labels the women $W_1, W_2, W_3, W_4, W_5, W_6$, and W'_j, $j = 1, \ldots 3\ell$. Also label the condoms C_1, C_2, C_3, C_4, C_5, C'_i, $i = 1, \ldots, k$, C''_j, $j = 1, \ldots, 2\ell$. The steps of the algorithm are:

Algorithm for m men, n women, $m/2 + 2n/3$ condoms:

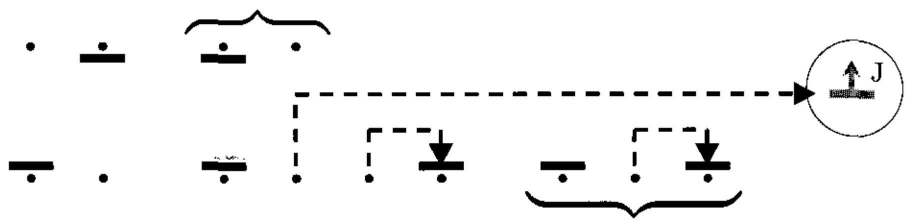
(a) M_2 will use condom C_1, M'_{2i-1} will use C'_i, $i = 1, \ldots, k$. W_1 will use C_2, and W_3, W_4, W_5 will use C_3, C_4, C_5 respectively. W'_{3j-2} will use C''_{2j-1}, $j = 1, \ldots, \ell$ and W'_{3j-1} will use C''_{2j}, $j = 1, \ldots, \ell$.



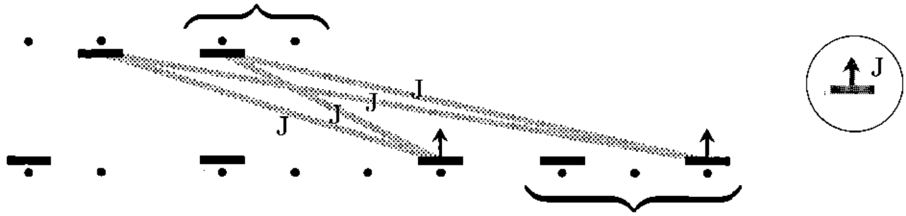
(b) All encounters M_2C_1 and $M'_{2i-1}C'_i$ with C_2W_1, C_3W_3, C_4W_4, C_5W_5, $C''_{2j-1}W'_{3j-2}$, $C''_{2j}W'_{3j-1}$.



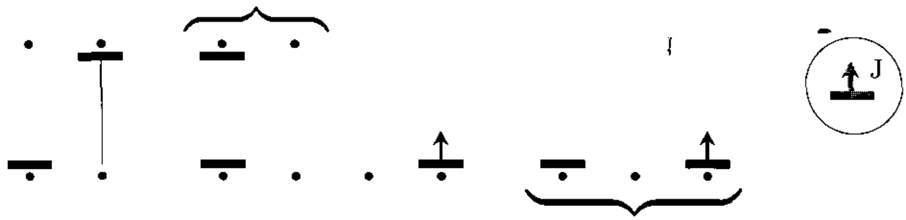
(c) W_4 gives up C_4 which will then be used as a master condom, and so will be denoted by J for the rest of the algorithm. W_5 passes C_5 to W_6. Each W'_{3j-1} passes C''_{2j} to W'_{2j}.



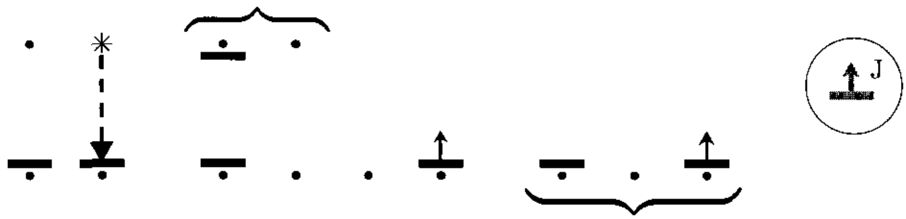
(d) Encounter $M_2C_1\overrightarrow{J}C_5W_6$, all encounters $M_2C_1\overrightarrow{J}C''_{2j}W'_{3j}$. All encounters $M'_{2i-1}C'_i\overrightarrow{J}C_5W_6$ and all encounters $M'_{2i-1}C'_i\overrightarrow{J}C''_{2j}W'_{3j}$.



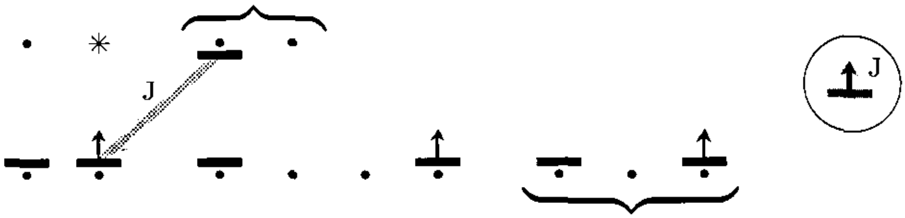
(e) M_2 has completed all encounters except W_2, so will no longer need C_1. One therefore has the encounter $M_2C_1W_2$.



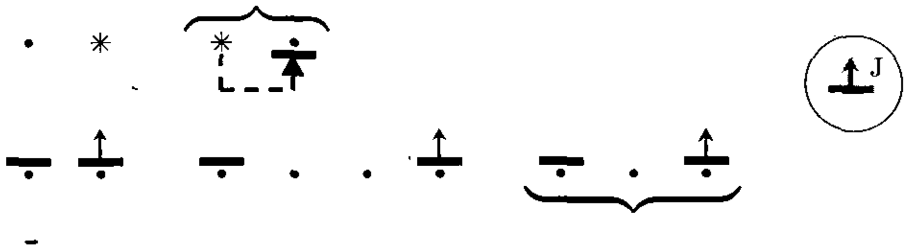
(f) M_2 now passes C_1 to W_2.



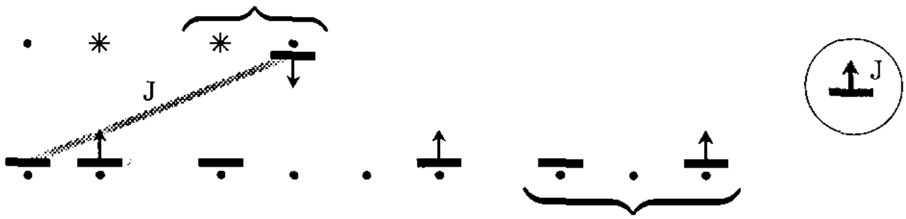
(g) All encounters $M'_{2i-1}C'_i\overrightarrow{J}C_1W_2$.



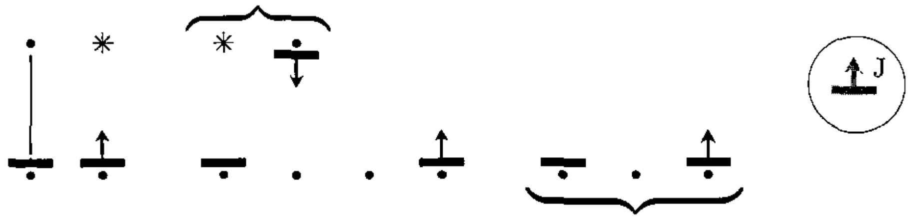
(h) Each M'_{2i-1} has completed all encounters so passes C'_i to M'_{2i}.



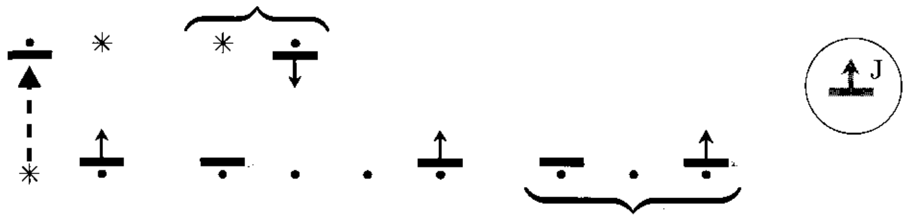
(i) All encounters $M'_{2i}C'_i\overleftarrow{J}C_2W_1$



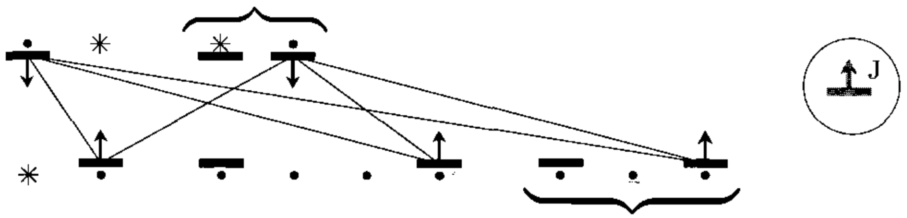
(j) W_1 has completed all encounters except with M_1 so that one has the encounter $M_1C_2W_1$.



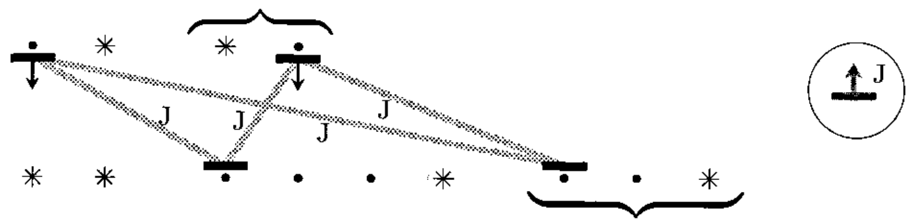
(k) W_1 passes C_2 to M_1.



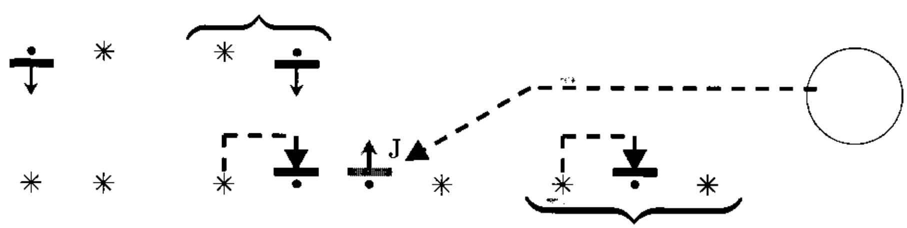
(l) Encounters M_1C_2 and $M'_{2i}C'_i$ with C_1W_2, C_5W_6, and all $C''_{2j}W_{3j}$.



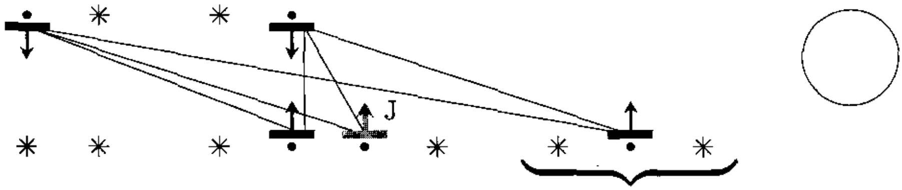
(m) Encounters M_1C_2 and $M'_{2i}C'_i$ with $\overleftarrow{J}\,C_3W_3$ and all $C''_{2j-1}W'_{3j-2}$.



(n) W_3 has finished so passes C_3 to W_4. W_5 uses J from now on. Each W'_{3j-2} has finished and passes C''_{2j-1} to W'_{3j-1}.

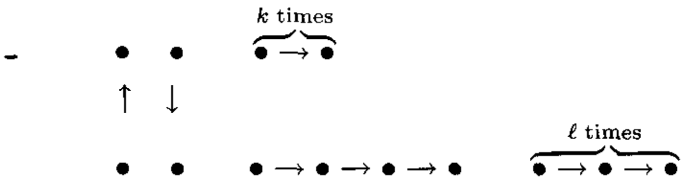


(o) All remaining encounters M_1C_2 and $M'_{2i}C'_i$ with C_3W_4, JW_5, and all $C''_{2j-1}W'_{3j-1}$.



Remark: *In the case of* $6n$ *men and* $6n$ *women this algorithm needs* $6n/2 + 2 \cdot 6n/3 = 7n$ condoms. By the lower bound of Section 10.1.2, *this algorithm is optimal.*

Here is a graph of the algorithm



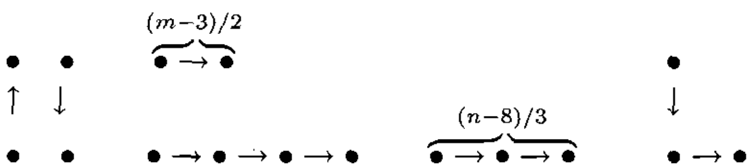
10.3 The general case

The algorithm of the last section can be modified to treat all the cases for $m, n \geq 6$. We only present the generic cases since the others can be handled similarly.

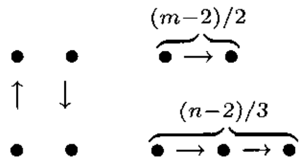
(a) $m \equiv 1 \pmod 2$ and $n \equiv 2 \pmod 3$

(b) $m \equiv 0 \pmod 2$ and $n \equiv 2 \pmod 3$

(a) We have an algorithm similar to the above based on the graph



(b) This case uses a different algorithm (similar to the Hajnal-Lovász method) using the graph



$J =$ master condom

where J serves as a "master condom," i.e., one side is always clean and the other always unclean, as before, but it never comes in contact *directly* with a M_i or W_j.

As in [55] I leave the case of $n < 6$ to the reader. Note, however, that the cases $m = n = 2$ and $n = 1$, $m = 2k + 1$, are the only ones for which (10.1) can be improved.

10.4 Lower bounds

To show that (10.1) is optimal, we must improve the lower bound of (10.3).

Lemma 1. *If $m \geq n \geq 6$, then (10.1) is optimal.*

(As before, I leave the case $n < 6$ to the reader.)

Proof: The only case when the lower bound of (1) doesn't follow directly from (10.3) is if

$$\left\lceil \frac{m}{2} + \frac{2n}{3} \right\rceil \neq \left\lceil \frac{m}{2} + \frac{2n}{3} - \frac{1}{3} \right\rceil$$

which happens when

(a) $m \equiv 0 \pmod 2$ and $n \equiv 2 \pmod 3$

(b) $m \equiv 1 \pmod 2$ and $n \equiv 1 \pmod 3$

We will prove the Lemma for (a). The proof for (b) is similar.

Proof for (a): By the comments at the end of §2 the graph of an algorithm taking $m/2 + 2n/3 - 1/3$ condoms must look like

$$(10.4) \qquad \begin{array}{cc} \bullet & \bullet \\ \uparrow & \downarrow \\ \bullet & \bullet \end{array} \qquad \begin{array}{c} \overbrace{\bullet \to \bullet}^{(m-2)/2} \\ \\ \overbrace{\bullet \longleftrightarrow \bullet \longleftrightarrow \bullet}^{(n-2)/3} \end{array}$$

I leave the proof for $n = 2$ to the reader. For $n > 2$ I will need the fact that the 3-components in (10.4) must all be of the form $\bullet \to \bullet \to \bullet$. As a matter of fact, the following stronger statement is true:

Lemma 2. *For any algorithm, if the men have a 2-component, then the 3-components of the women must be of the form* $\bullet \to \bullet \to \bullet$.

For the moment assume that Lemma 2 is true, then the graph (10.4) looks like

$$\begin{array}{cc} \bullet & \bullet \\ \uparrow & \downarrow \\ \bullet & \bullet \end{array} \qquad \begin{array}{c} \overbrace{\bullet \to \bullet}^{(m-2)/2} \\ \\ \overbrace{\bullet \to \bullet \to \bullet}^{(n-2)/3} \end{array}$$

Now consider the last $W_{3i+5}M_3$ for $0 \le i \le (n-5)/3$, say it is $W_{3j+5}M_3$. M_3 has to be protected, but I will show that no intermediate condoms can be used: It is fairly clear that $\overrightarrow{W_1M_1}$ and $\overrightarrow{M_2W_2}$ can't be used. None of the $\overrightarrow{W_{3i+3}W_{3i+4}}$ can be used since one side is reserved for $W_{3i+3}M_4$ and the other is reserved for $W_{3i+4}M_4$. Finally, by assumption all other $\overrightarrow{W_{3i+4}W_{3i+4}}$ are unclean on both sides. Therefore $W_{3j+5}M_3$ is impossible.

Proof of Lemma 2: Let $M_1 \to M_2$ be a 2-component of the men, and $W_1 \leftrightarrow W_2 \leftrightarrow W_3$ be a 3-component of the women. The possibilities for the 3-component that have to be ruled out are:

(i) $\bullet \leftarrow \bullet \to \bullet$

(ii) $\bullet \to \bullet \leftarrow \bullet$

We consider the case of (i) (the (ii) case is handled similarly).

We try to figure out in which order W_1, W_2, W_3 will have sex with M_2. Without loss of generality, one can assume that $\overrightarrow{W_2W_3}$ will be used for W_2M_2. Now by definition of the directed graph $\overrightarrow{M_1M_2}$ is unclean on both sides. Furthermore, since M_1 has completed all its encounters, $\overrightarrow{W_2W_1}$ and $\overrightarrow{W_2W_3}$ are unclean on both sides (this follows from the shape of the component).

(a) W_2 cannot be first since this would mean that the W_3 side of $\overrightarrow{W_2W_3}$ would have to be protected, which is impossible, since a new condom would become infected by W_3, and W_3 only has one edge incident to it.

(b) W_3 cannot be first since the W_2 side of $\overrightarrow{W_2W_3}$ would have to be protected. From the assumption about the 3-component, this means that $\overrightarrow{W_2W_1}$ would have to be put over $\overrightarrow{W_2W_3}$. But the W_1 side of $\overrightarrow{W_2W_1}$ would have to be protected, which is impossible as before.

(c) Using the same argument as in (b) it is seen that W_1 cannot be first.

This means that none of W_1, W_2, W_3 can be first and the lemma follows by contradiction.

Exercise 10.1 Show that a formula similar to (10.1) holds for the case where all pairs of individuals have a sexual encounter.

Exercise 10.2 [M] (A. Orlitzky and L. Shepp [93]) Analyze the condom problem with m men, n women, $m \geq n$ where all men are bisexual.

Exercise 10.3* [M] In general consider the condom problem where a *preference graph* is given and two people have a sexual encounter if and only if they are joined by an edge (thus the original problem corresponds to the complete bipartite graph).